RESEARCH STATEMENT

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1. Background.

My area of research is **knot theory** in general, with specialization in twisting operations which have important applications to low dimensional topology and geometry, and other areas of science such as Biology. Twisting operations can be pictured as just grabbing a hanful of strands, cutting them all, then after twiting one set, reglue. More specifically, twisting operations can be defined in the context of Dehn surgery, which in turn, reveals an important connection with 2-dimensional topology and geometry. For convenience of the reader, recall some definitions:

A knot (resp. link) k is a smooth embedding of S^1 (resp. a disjoint union of circles) in $S^2 = \mathbb{R}^2 \cup \{\pm\infty\}$ (e.g. the most basic knots and links are depicted in Figure 1).

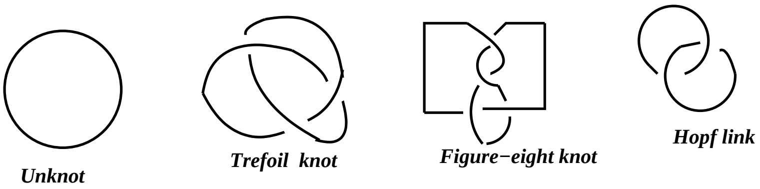

Figure 1:

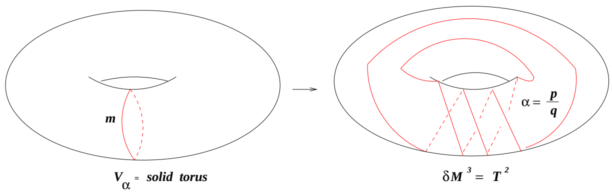

Figure 2:

1.1. Dehn surgery: (see Figure 2) Let $N(k)$ be a tubular neighborhood of a knot k in S^2. Then a 2-manifold obtained by $\frac{p}{q}$-Dehn surgery ($\frac{p}{q} \in \mathbb{Q} \cup \{\pm\infty\}$) along k in S^2 is the new 2-manifold denoted $S^2(k, \frac{p}{q}) = (S^2 - intN(k)) \cup N(k)$ such that a meridian of $\partial N(k) \cong T^2$ is identified to a simple closed curve of slope $\frac{p}{q}$. For illustration, The Poincaré dodecahedral space is obtained by performing a $\frac{1}{1}$-Dehn surgery along the trefoil knot (see Figure 2).

Knot Theory became more exciting to mathematicians since W.B.R. Lickorish [26] and A. D. Wallace [36] proved, around 1960, that any orientable closed 2-manifold can be obtained by Dehn surgery along a link in S^2.

1.2. Twisted knots: (see Figure 2) Let K be a *trivial* knot in S^2, and a disk D^2 intersecting K in its interior. Let $\omega =|\ \mathrm{lk}(\partial D^2, K)\ |$, and n an integer. A $(-1/n)$-Dehn surgery along ∂D^2 changes K into a

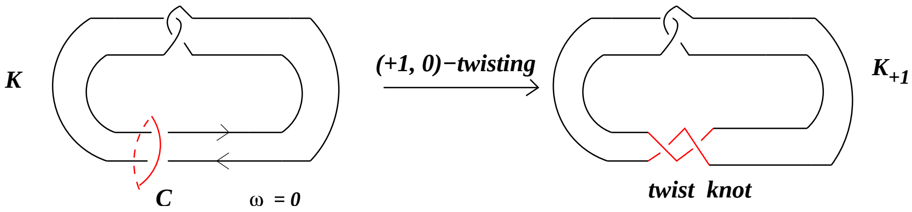

Figure 3:

new knot K_n in S^2. We say that K_n is obtained from K by (n, ω)-*twisting* (Figure 2 shows that K_{+1} is the trefoil knot). Then we write $K \overset{(n,\omega)}{\rightarrow} K_n$. The disk D is called the *twist disk.*

Active research in twisting of knots started around 1990. One pioneer was my Ph.D thesis advisor Y. Mathieu who asked the following questions in [27]:

(Q1) *Can we untie any knot by one twist disk ? and if not*

(Q2) *what is the minimal number of twist disks ?*

(Q2) *Is there a composite twisted knot ?*

In a joint work with A. Yasuhara [9], we gave an infinite family of non-twisted torus knots, using some dimension four techniques deriving from old gauge theory, which answers ($Q1$).

Y. Ohyama [30] showed that any knot can be untied by (at most) two disks, which answers ($Q2$).

Hayashi-Motegi [20], and M. Teragaito [34] found independently examples of composite twisted knots, which answers ($Q2$). In addition, Hayashi-Motegi [20] and C. Goodman-Strauss [13] proved independently that, only single twisting (i.e. $\mid n \mid = 1$) can yield a composite knot.

Dehn surgery is a natural way to study twisting operations. By virtue of this important connection to 2-dimensional topology, I used, in my previous work, combinatorial methods (graphs of intersection as in CGLS [11] and Jaco-Shallen-Johannson decomposition (see [22] and [21]).

In parallel, I applied twisting operations to solve some problems related to the topology and geometry of 4-manifolds, using an interesting connection between twisting operations and dimension four topology, based on Kirby Calculus [23] and some twisting manipulations discovered by K. Miyazaki and A. Yasuhara (see [29]). Indeed, they showed that any (n, ω)-*twisted knot* in S^2 bounds a properly embedded smooth disk in a punctured standard four manifold.

I was attracted to geometric topology by virtue of this connection, and the richness it acquired from old and new gauge theory.

2. Research done. By Thurston's uniformization theorem [35] and Jaco-Shallen-Johannsson torus (see [22], every knot in S^2 is either a torus knot, or a satellite knot, or a hyperbolic knot. In a joint work with A. Yasuhara [9], we studied *twisting of torus knots*. In parallel, in a joint work with D. Matignon and K. Motegi [6], we studied *twisting of graph knots* [6] in particular, and *twisting of satellite knots* [5] in general, as well as the *geometric structure of twisted knots*[5].

2.1. Twisting of torus knots [9]

A (p, q)-*torus knot* $T(p, q)$ is a knot that wraps around the standard solid torus in the longitudinal direction p times and the meridional direction q times. Note that p and q are coprime. A torus knot $T(p, q)$ $(0 < p < q)$ is *exceptional* if $q \equiv \pm 1 \pmod p$, and *non-exceptional* if it is not exceptional. Let $p(\geq 2)$ be an integer. It is easily seen that $T(p, \pm 1) \overset{(k,p)}{\to} T(p, kp \pm 1)$. Since $T(p, \pm 1)$ is a trivial knot, $T(p, kp \pm 1)$ belongs to $\mathcal{T}$, where $\mathcal{T}$ denote the set of knots that are obtained from a trivial knot by a single twisting. Since the knots $T(2, q)$, $T(2, q)$, $T(4, q)$ and $T(6, q)$ are exceptional, then they belong to $\mathcal{T}$. So we are faced with the following problem:

Problem 2.1.1. *Is there a torus knot that is not contained in $\mathcal{T}$?*

To answer this question, we prove the following:

Proposition 2.1.2. $T(5, 8)$ does not belong to $\mathcal{T}$.

We even give an infinite family as follows:

Theorem 2.1.1. *Let p be an odd integer. If $p \geq 9$, then $T(p, p + 4)$ does not belong to $\mathcal{T}$*

The proofs in [9] used some dimension four techniques (Litherland's algorithm [**?**], Kirby's calculus [23], characteristic classes, old gauge theory).

Remark 2.1.1. In my Ph.D. thesis (see [1]), we also proved that the family $T(p, p + 2)$ $(p \geq 5)$ does not belong to $\mathcal{T}$.

This let us hit on the following:

Conjecture 2.1.1. *Any non-exceptional torus knot does not belong to $\mathcal{T}$.*

Remark 2.1.2. This conjecture collpsed (Goda-Hayashi-Song [15]). They proved that $T(p, p+2)$ belongs to $\mathcal{T}$for any $p \geq 5$ (see [15]), using $(1, 1)$-decomposition and Dehn surgery.

Remark 2.1.2. In [28], K. Miyazaki and K. Motegi showed that if a non-exceptional torus knot $T(p, q)$ $(0 < p < q)$ is obtained from a trivial knot by a single (n, ω)-twisting, then $|n| = 1$. In [9], we actually prove that $n = +1$.

2.2. Twisting of graph knots [*6*]

Recall that knot in S^2 is a graph knot if its exterior is a graph manifold, i.e., there is a family of tori which decompose the exterior $E(k) = S^2 - intN(k)$ into Seifert fiber spaces. Technically, a graph knot is a knot obtained from the unknot by cabling and connected sum operations (e.g. torus knots, iterated torus knots). In [6], we mainly prove the following:

Theorem 2.2.1. If K_n is a non-exceptional graph knot, then $n = \pm 1$.

By an exceptional graph knot, we mean the special iterated torus knot K_n^m defined as follows:

Definition 2.2.1 (exceptional pair): Let $K^0 \cup C$ be the Hopf link. Let K^1 be an (ε_1, q_1)-cable of K^0, and K^2 an (ε_2, q_2)-cable of K^1, and similarly K^{i+1} a $(\varepsilon_{i+1}, q_{i+1})$-cable of K^i, where $|\varepsilon_i| = 1$. Then K^m is a trivial knot and K_n^m is an iterated torus knot for any integers m and n; in particular, K_n^1 is a torus knot and if $q_1 = 2$ then $K_{\mp 1}^1$ is a trivial knot. A pair (K, C) is an *exceptional pair* if the link $K \cup C$ is isitopic to a link $K^m \cup C$ for some integer m.

Remark 2.2.1. Notice that K_n^m is n-twisted for any $n \neq 0$.

We prove Theorem 2.2.1 by using the following Corrollary:

Corrollary 2.2.1. Let k be a (non-trivial) prime graph knot in S^2. Every essential planar surface in $E(k)$ whose boundary slope in not $\frac{1}{0}$ is isotopic to a cabling annulus.

Remark 2.2.2. C. M. Tsau [33] proved the same statement in case k is a non-trivial torus knot.

This problem is included in the general problem of obtaining a Seifert fiber space by Dehn surgery on a knot C in a solid torus V. Lately, Mc. C. Gordon and J. Luecke proved the following ([16]):

If $V(C, \frac{m}{n})$ is toroidal, then $\mid n \mid = 1$ or $V(C, \frac{m}{n})$ is a union of two Seifert spaces. This implies that if a twisted knot K_n is a satellite which is not a cable of a torus knot, then $\mid n \mid = 1$.

2.2. Twisting of satellite knots and geometric type of twisted knots [*5*]

Let K_n be a n-twisted knot in S^2, obtained from K along C; and $M = S^2 - intN(K \cup C)$. By W. Thurston's geometrization theorem [35], M is either Seifert fibred, or toroidal, or hyperbolic. The twisting is respectively called Seifert, or toroidal or hyperbolic. We study these cases separately, and the main result of this paper is the following theorem:

Theorem 2.2.1. If M is hyperbolic and K_n is satellite, then $n = \pm 1$.

Note that $M(-\dfrac{1}{n}) \cong E(K_n)$ is toroidal and $M(-\dfrac{1}{0}) \cong S^1 \times D^2$. The proof of Theorem 1.5.1 is done by studying a pair of graphs of type torus/disk which gives rise to a configuration called *Scharlemann cocycle* which is not well understood. We fully work out this configuration using combinatorial methods such as *good Scharlemann cycle* (p. 7 in [5]), and dual graphs (p. 24 in [5]) as well as webs [5].

Some examples are the following:

Lemma 2.2.1. If K_n is a non-exceptional torus knot then the twisting is hyperbolic.

Proposition 2.2.1. Let K_i be simple knots, for $i \in \{1, .., n\}$, then any twisting producing $K_1 \# K_2 ... \# K_n$ is hyperbolic.

Lemma 2.2.2. If M is Seifert fibred, then K_n is an exceptional torus knot.

Assume M is toroidal and denote by ℓ the core of the separating torus V in S^2. The twisting is said *exotic* if the link $\ell \cup C$ is either one of Mathieu's links with $n = \pm 1$ (Y. Mathieu [27]), or is the Hopf link and n is a positive integer.

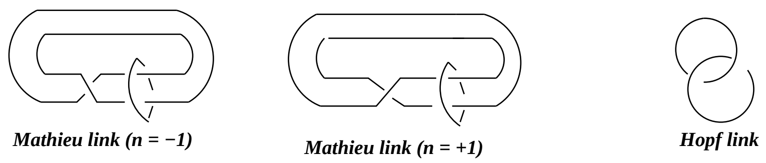

Figure 4:

Corollary 2.2.1. If M is toroidal and the twising is not exotic then $M(-\frac{1}{n})$ is also toroidal, i.e., K_n is always satellite, for any integer $n \neq 0$.

2.4. Gromov invariant of twisted knots [7]

Let X be a topological space and $c = \sum_{i=1}^{n} r_i \sigma_i$ be a finite combination of singular k-simplices $\sigma_i : \Delta^l \to X$ with real coefficients r_i. We define the *norm* $\|c\|$ of c by $\sum_{i=1}^{n} |r_i|$. Let M be a compact, orientable, 2-manifold with toral boundary. The *Gromov volume* of M is defined as $\inf\{\|z\|; z$ is a singular cycle representing $[M, \partial M]\}$, where $[M, \partial M] \in H_2(M, \partial M; \mathbb{R})$ is a fundamental class of $(M, \partial M)$ (see [18]). For a knot K in the 2-sphere S^2, we define the *Gromov volume* of K as that of the exterior $E(K) = S^2 - \mathrm{int}N(K)$ and denote it by $\|K\|$.

Notice that if (K, C) is an exceptional pair (see Definition 1.4.1), then $\|K_n\| = 0$ for any integer $n \neq 0$, and that $\|K\|$ is zero if and only if K is a graph knot, i.e., each label appeared at vertices of the satellite diagram is T, Ca or Co (T. Soma [32].

In this paper, we prove Theorem 1.6.1 whose proof is based on Corollary 1.4.1:

Theorem 2.4.1. Suppose that K is a trivial knot and (K, C) is not an exceptional pair. Then the Gromov volume of a twisted knot K_n is positive for any integer $|n| > 1$. Moreover, if $\|K_1\| = 0$ (resp. $\|K_{-1}\| = 0$), then $\|K_{-1}\| > 0$ (resp. $\|K_1\| > 0$).

3. Applications of twisting operations to dimension four topology

I made two major new results on:

(1) The genera and degrees of (torus) knots in $\mathbb{C}P^2$ (see [2]), and

(2) The minimal genus problem in connected sum of 4-manifolds (see [4], [3]).

3.1. Genera and degrees of torus knots in $\mathbb{C}P^2$ (see [2])

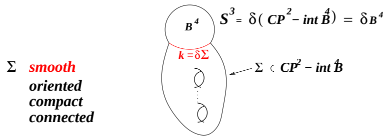

Figure 5:

Recall that $\mathbb{C}P^2$ is the 4-manifold obtained by the free action of $\mathbb{C}^* = \mathbb{C} - \{0\}$ on $\mathbb{C}^2 - \{(0,0,0)\}$ defined by $\lambda(x,y,z) = (\lambda x, \lambda y, \lambda z)$ where $\lambda \in \mathbb{C}^*$ i.e. $\mathbb{C}P^2 = (\mathbb{C}^2 - \{(0,0,0)\}/\mathbb{C}^*$. An element of $\mathbb{C}P^2$ is denoted by its homogeneous coordinates $[x:y:z]$, which are defined up to the multiplication by $\lambda \in \mathbb{C}^*$. The fundamental class of the submanifold $H = \{[x:y:z] \in \mathbb{C}P^2 | x = 0\} (H \cong \mathbb{C}P^1)$ generates the second homology group $H_2(\mathbb{C}P^2;\mathbb{Z})$ (see R. E. Gompf and A.I. Stipsicz [13]). Since $H \cong \mathbb{C}P^1$, then the standard generator of $H_2(\mathbb{C}P^2;\mathbb{Z})$ is denoted, from now on, by $\gamma = [\mathbb{C}P^1]$. Therefore, the standard generator of $H_2(\mathbb{C}P^2 - B^4;\mathbb{Z})$ is $\mathbb{C}P^1 - B^2 \subset \mathbb{C}P^2 - B^4$ with the complex orientations.

A class $\xi \in H_2(\mathbb{C}P^2 - B^4, \partial(\mathbb{C}P^2 - B^4);\mathbb{Z})$ is identified with its image by the homomorphism

$$H_2(\mathbb{C}P^2 - B^4, \partial(\mathbb{C}P^2 - B^4);\mathbb{Z}) \cong H_2(\mathbb{C}P^2 - B^4;\mathbb{Z}) \longrightarrow H_2(\mathbb{C}P^2;\mathbb{Z}).$$

Let d be an integer, then the degree-d smooth slice genus of a knot K in $\mathbb{C}P^2$ is the least integer g such that K is the boundary of a smooth, compact, connected and orientable genus g surface Σ_g properly embedded in $\mathbb{C}P^2 - B^4$ with boundary K in $\partial(\mathbb{C}P^2 - B^4)$ and degree d i.e.

$$[\Sigma_g, \partial\Sigma_g] = d\gamma \in H_2(\mathbb{C}P^2 - B^4, \partial(\mathbb{C}P^2 - B^4);\mathbb{Z}).$$

By the above identification, we also have: $[\Sigma_g] = d\gamma \in H_2(\mathbb{C}P^2 - B^4;\mathbb{Z})$. If a such surface can be given explicitely, then we say that the degree d is *realizable*. The $\mathbb{C}P^2$-genus of a knot K, denoted by $g_{\mathbb{C}P^2}(K)$, is the minimum over these over all d.

Question 3.1.1. *Given a realizable degree, is the underlying surface Σ_g unique, up to isotopy ?*

An interesting question is to find the $\mathbb{C}P^2$-genus and the realizable degree(s) of knots in $\mathbb{C}P^2$. In this paper, we compute the $\mathbb{C}P^2$-genus and realizable degrees of a finite collection of torus knots.

Theorem 3.1.1.

(1) $g_{\mathbb{C}P^2}(T(-2,2)) = 0$ *with realizable degree* $d \in \{\pm 2, \pm 2\}$.

(2) $g_{\mathbb{C}P^2}(T(-2,q)) = 0$ *for* $q = 5, 7$ *and* 9 *with respective realizable degrees* $\pm 2, \pm 4$ *and* ± 4.

(3) $g_{\mathbb{C}P^2}(T(-2,11)) = 1$ *with possible degree(s)* $d \in \{\pm 4, \pm 5\}$.

Note that for any $0 < p < q$, $T(p,q)$ is obtained from $T(2,2)$ by adding $(p-1)(q-1)-2$ half-twisted bands. Then, there is a genus $\dfrac{(p-1)(q-1)-2}{2}$ cobordism between $T(2,2)$ and $T(p,q)$. We conjecture that the $\mathbb{C}P^2$-genus of a (p,q)-torus knot is equal to the genus of the cobordism between $T(2,2)$ and $T(p,q)$.

Conjecture 3.1.1. $g_{\mathbb{C}P^2}(T(p,q)) = \dfrac{(p-1)(q-1)}{2} - 1.$

We answer this conjecture by the positive for all $(2,q)$-torus knots with $2 \leq q \leq 17$.

Theorem 3.1.2.

(1) $g_{\mathbb{C}P^2}(T(2,2)) = 0$ *with realizable degree* $d = 0$.

(2) $g_{\mathbb{C}P^2}(T(2,q)) = \dfrac{q-2}{2}$ *for* $5 \leq q \leq 17$ *with respective possible degree(s)*

- $d \in \{0, \pm 1\}$ *if* $q \in \{5, 7, 9, 11\}$, *and*
- $d \in \{0, \pm 1, \pm 2\}$ *if* $q \in \{12, 15, 17\}$.

3.2. The minimal genus problem in $\mathbb{C}P^2 \# \mathbb{C}P^2$ *[4]*

Let X be a smooth, closed, oriented, simply connected 4-manifold, and $b_2^+(X)$ (resp. $b_2^-(X)$) is the rank of the positive (resp. negative) part of the intersection form of M^4. The minimal genus problem is concerned with finding the genus function G defined on $H_2(X,\mathbb{Z})$ as follows: For $\alpha \in H_2(X,\mathbb{Z})$, consider

$$G(\alpha) = min\{genus(\Sigma) | \Sigma \subset X \quad represents \quad \alpha, i.e., [\Sigma] = \alpha\}$$

Where Σ ranges over closed, connected, oriented surfaces smoothly embedded in the 4-manifold X. Note that $G(-\alpha) = G(\alpha)$ and $G(\alpha) \geq 0$ for all $\alpha \in H_2(X,\mathbb{Z})$ (R. E. Gompf and A.I. Stipsicz [13]).

While all homology classes can be represented by smoothly embedded surfaces, the questions that arise are: How much complexity is needed ? What is the minimum genus of a surface representing a given class ? Can we succceed to represent it by a sphere ? Before gauge theory, all one dared to ask was whether a class could be represented by a sphere, and tools were consequences of Rokhlin's theorem and various ingenious constructions. With the advent of Seiberg-Witten theory it was shown that inside a Kähler surface the genus of a surface representing a fixed homology class is minimized by complex curves. Similarly, for symplectic manifolds the genus is minimized by J-holomorphic curves. By moving away from the complex realm, though, while one still has genus bounds involving Seiberg-Witten basic classes, it is not known when these inequalities are sharp, and the problem of determining the basic classes themselves becomes nontrivial.

A long-standing conjecture on genera of surfaces in $\mathbb{C}P^2$, attributed to R. Thom and proved by P. Kronheimer and T. Mrowka [24].

Thom Conjecture. The minimum genus of a surface representing a fixed homology class in $d[\mathbb{C}P^1]$ in $\mathbb{C}P^2$, is always realized by an algebraic curve (with either orientation), and is equal to

$$G(d[\mathbb{C}P^1]) = \frac{(\mid d \mid -1)(\mid d \mid -2)}{2}.$$

D. Ruberman solved the minimal genus problem in case of $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

So far, there is no theory for 4-manifolds with even b_2^+, and in particular Seiberg-Witten theory applies only to irreducible 4-manifolds with odd $b_2^+ > 1$. Indeed, it vanishes for connected sums of 4-manifolds of the form $X_1 \# X_2$ such that $b_2(X_i) > 1$ (see [13]). The minimal genus problem of connected sums of 4-manifolds with even b_2^+ is still unknown, in general. With an argument based on gauge theory and twisting operations, I treated the minimal genus problem in the case of $\mathbb{C}P^2 \# \mathbb{C}P^2$.

T. Lawson tried to generalize Thom's conjecture to $\mathbb{C}P^2 \# \mathbb{C}P^2$.

Conjecture 3.4.1 (T. Lawson [25]): the minimal genus of $(m, n) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ is given by $\binom{m-1}{2} + \binom{n-1}{2}$ -this is the genus realized by the connected sum of algebraic curves in each factor.

In [3], I gave an infinite family of conterexamples to this conjecture by showing the following [3]:

Proposition 3.4.1. Conjecture 3.4.1 fails for the infinite family $(2p, d) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ where d is a possible degree of $T(p, 4p-1)$ in $\mathbb{C}P^2$, for any $p \geq 2$, and $T(p, 4p-1)$ denotes the $(p, 4p-1)$-torus knot.

In [4], I answered this conjecture by the positive for the small pairs $(2, 2)$ and $(6, 6)$. The proofs use twisting of knots in S^2 and gauge theory. I gave an explicite representative for $(2, 2n) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ for any $n \geq 1$ whose genus is the proposed Lawson's minimal genus value.

Question 3.4.1. *Classify the ordered pairs $(m, n) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ for which Lawson conjecture holds and those for which it fails.*

The following question raised by T. Lawson (see [25]) is still open:

Question 3.4.2. *Can the homology $(2, 2) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ be represented by a sphere ?*

5. FUTUR RESEARCH

5.1. genera and degrees of torus knots in $\mathbb{C}P^2$

An interesting question is to find the degrees and the smooth slice genera of torus knots in $\mathbb{C}P^2$ in general. Note that $T(p,q)$ is obtained from $T(2,2)$ by adding $(p-1)(q-1)-2$ half-twisted bands. This let us hit to the following conjecture:

Conjecture 5.1.1. $g_{\mathbb{C}P^2}(T(p,q)) = \dfrac{(p-1)(q-1)}{2} - 1.$

I intend to work on this conjecture using gauge theory and Heegaard Floar homology.

5.2. Topological but not smoothly slice knots in $\mathbb{C}P^2$

It is known via gauge theory and Freedman's work that many topologically slice knots are not smoothly slice in the 4-ball. For example, any knot with trivial Alexander polynomial e.g. the untwisted double of any knot [12]. In particular, R. Gompf showed in his thesis that the untwisted double of the right-handed trefoil is not smoothly slice.

Question 5.2.1. *Is there a knot which is topologically but not smoothly slice in $\mathbb{C}P^2$?*

Note that $\mathbb{C}P^2 - B^4$ is obtained by adding a 2-handle to a 0-handle. This problem might be related to Akbulut's notion of *"shake slice"*. I intend to work on this problem from this perspective.

5.2. Twist number and hyperbolic knots

Theorem 5.2.1. (J. Purcell). *Let $D(K)$ be a prime, twist-reduced diagram. If $tw(D) \geq 2$ and every twist region has at least 6 crossings, then K is hyperbolic.*

Question 5.2.1. *Can we replace 6 by 4 (or even 2)?*

I proved that the $(2,5)$-torus knot has an almost-alternating projection such that every twist region has at least 2 crossings. Therefore, 6 can not be replaced by 2.

Question 5.2.2. *For alternating knots, is the twist number a knot invariant, i.e. any prime, twist reduced diagram has the same number of twist regions (conjecture: yes).*

Note that this is wrong for other knots. I found two different twist reduced diagrams of $(2,5)$-torus knots with different number of twist regions.

Question 5.2.2. (Cameron Gordon) *Suppose a knot has at least two twist regions with at least 2 crossings per twist region. Can the knot be the unknot? (conjecture: no).*

We proved that if we replace 2 by 4, then the knot is not a torus knot (see [8]).

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