Twisting of Composite Torus Knots

MOHAMED AIT NOUH

Abstract. We prove that the family of connected sums of torus knots $T(2, p) \# T(2, q) \# T(2, r)$ is nontwisted for any odd positive integers $p, q, r \geq 3$, partially answering in the positive a conjecture of Tera-gaito [22].

1. Introduction

Let $K$ be a knot in the 3-sphere $S^3$, and $D^2$ a disk intersecting $K$ in its interior. Let $n$ be an integer. A $(-\frac{1}{n})$-Dehn surgery along $C = \partial D^2$ changes $K$ into a new knot $K_n$ in $S^3$. Let $\omega = \text{lk}(\partial D^2, L)$. We say that $K_n$ is obtained from $K$ by $(n, \omega)$-twisting (or simply twisting). Then we write $K \to^{(n,\omega)} K_n$ or $K \to^{(n,\omega)} (n, \omega)$. We say that $K_n$ is an $(n, \omega)$-twisted knot (or simply a twisted knot) if $K$ is the unknot (see Figure 1).

An easy example is depicted in Figure 2, where we show that the right-handed trefoil $T(2, 3)$ is obtained from the unknot $T(2, 1)$ by a $(+1, 2)$-twisting (in this case, $n = +1$ and $\omega = +2$). A less obvious example is given in Figure 3, where it is shown that the composite knot $T(2, 3) \# T(2, 5)$ can be obtained from the unknot by a $(+1, 4)$-twisting (in this case, $n = +1$ and $\omega = +4$); see [13]. Here, $T(2, q)$ denotes the $(2, q)$-torus knot (see [14]).

Active research on twisting of knots started around 1990. One pioneer was the author’s Ph.D. thesis advisor Y. Mathieu, who asked the following questions in [16].

Question 1.1. Is every knot in $S^3$ twisted? If not, what is the minimal number of twisting disks?

Question 1.2. Is every twisted knot in $S^3$ prime?

To answer Question 1.1, Miyazaki and Yasuhara [18] were the first to give an infinite family of knots that are nontwisted. In particular, they showed that the granny knot, that is, the product of two right-handed trefoil knots, is the smallest nontwisted knot. In his Ph.D. thesis [3], the author showed that $T(5, 8)$ is the smallest nontwisted torus knot. This was followed by a joint work with Yasuhara [5], in which we gave an infinite family of nontwisted torus knots (i.e., $T(p, p+7)$ for any $p \geq 7$) using some techniques derived from old gauge theory. On the other hand, Ohyama [19] showed that any knot in $S^3$ can be untied by (at most) two disks.

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To answer Question 1.2, Hayashi and Motegi [13] and M. Teragaito [23] independently found examples of composite twisted knots (see Figure 3). In particular, Goodman-Strauss [11] showed that any composite knot of the form $T(p, q) \# T(-q, p + q)$ is a twisted knot for any coprime positive integers $1 < p < q$. More generally, Hayashi and Motegi [13] and Goodman-Strauss [11] proved independently that only single twisting (i.e., $|n| = 1$) can yield a composite knot. The tools used were combinatorial methods as in CGLS [8]. Moreover, Goodman-Strauss [11] proved that $K_1$ and $K_{-1}$ cannot both be composite and classified all composite knots of the form $K_1 \# K_2$, where $K_1$ and $K_2$ are both prime knots (for an extensive list of twisted composite knots, we refer the reader to the appendix of Goodman-Strauss’s paper [11]). However, there is no known twisted knot with three or more factors, that is, $k = k_1 \# k_2 \# \cdots \# k_m$, where $k_i$
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1. Conjecture 1.1 (Teragaito [22]). Any composite knot with three or more factors is nontwisted.

In this paper, we prove the following theorem.

**Theorem 1.1.** $T(2, p) \# T(2, q) \# T(2, r)$ is not twisted for any odd positive integers $p, q, r \geq 3$.

2. Preliminaries

In what follows, let $X$ be a smooth, closed, oriented, simply connected 4-manifold. Then the second homology group $H_2(X; \mathbb{Z})$ is finitely generated (for details, we refer to the book by Milnor and Stasheff [17]). The ordinary form $q_X : H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \rightarrow \mathbb{Z}$ given by the intersection pairing for 2-cycles such that $q_X(\alpha, \beta) = \alpha \cdot \beta$ is a symmetric unimodular bilinear form. The signature of this form, denoted $\sigma(X)$, is the difference of the numbers of positive and negative eigenvalues of a matrix representing $q_X$. Let $b_2^+(X)$ (resp. $b_2^-(X)$) be the rank of the positive (resp. negative) part of the intersection form of $X$. The second Betti number is $b_2(X) = b_2^+(X) + b_2^-(X)$, and the signature is $\sigma(X) = b_2^+(X) - b_2^-(X)$. From now on, a homology class in $H_2(X - B^4, \partial; \mathbb{Z})$ is identified with its image by the homomorphism

$$H_2(X - B^4, \partial(X - B^4); \mathbb{Z}) \cong H_2(X - B^4; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z}).$$

Recall that $\mathbb{C}P^2$ is the closed 4-manifold obtained by the free action of $\mathbb{C}^* \equiv \mathbb{C} - \{0\}$ on $\mathbb{C}^3 - \{(0, 0, 0)\}$ defined by $\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$, where $\lambda \in \mathbb{C}^*$, that is, $\mathbb{C}P^2 = (\mathbb{C}^3 - \{(0, 0, 0)\})/\mathbb{C}^*$. An element of $\mathbb{C}P^2$ is denoted by its homogeneous coordinates $[x : y : z]$, which are defined up to the multiplication by $\lambda \in \mathbb{C}^*$. The fundamental class of the submanifold $H = \{[x : y : z] \in \mathbb{C}P^2 | x = 0\}$ ($H \cong \mathbb{C}P^1$) generates the second homology group $H_2(\mathbb{C}P^2; \mathbb{Z})$ (see Gompf and Stipsicz [11]). Since $H \cong \mathbb{C}P^1$, the standard generator of $H_2(\mathbb{C}P^2; \mathbb{Z})$ is denoted, from now on, by $y = [\mathbb{C}P^1]$. Therefore, the standard generator of $H_2(\mathbb{C}P^2 - B^4, \mathbb{Z})$ is $\mathbb{C}P^1 - B^2 \subset \mathbb{C}P^2 - B^4$ with complex orientations.

Let $\alpha = S^2 \times \{\ast\}$ and $\beta = \{\ast\} \times S^2$ denote the standard generators of $H_2(S^2 \times S^2; \mathbb{Z})$ such that $\alpha^2 = \beta^2 = 0, \alpha \cdot \beta = 1$, and let $\gamma$ (resp. $\gamma'$) be the standard generators of $H_2(\mathbb{C}P^2; \mathbb{Z})$ (resp. $H_2(\overline{\mathbb{C}P}^2; \mathbb{Z})$) with $\gamma^2 = +1$ (resp. $\gamma'^2 = -1$).

A second homology class $\xi \in H_2(X; \mathbb{Z})$ is said to be characteristic if $\xi$ is dual to the second Stiefel–Whitney class $w_2(X)$ or, equivalently,

$$\xi \cdot x \equiv x \cdot x \pmod{2}$$

for any $x \in H_2(X; \mathbb{Z})$ (we leave the details to Milnor and Stasheff [17]).
Example 2.1. \((a, b) \in H_2(S^2 \times S^2; \mathbb{Z})\) is characteristic if and only if \(a\) and \(b\) are both even.

Example 2.2. \(d\gamma \in H_2(\mathbb{CP}^2; \mathbb{Z})\) is characteristic if and only if \(d\) is odd.

The following theorems give obstructions on the genus of an embedded surface representing either a characteristic class or bounding a knot in a punctured 4-manifold. Recall that the Arf invariant of a knot \(K\) is denoted by \(\text{Arf}(K)\), \(\sigma_p(K)\) denotes the Tristram \(p\)-signature [24], and \(e_2(K)\) denotes the minimum number of generators of \(H_2(X_K; \mathbb{Z})\), where \(X_K\) is the 2-fold branched covering of \(S^3\) along \(K\).

Theorem 2.1 (Acosta [1]). Suppose that \(\xi\) is a characteristic homology class in an indefinite smooth oriented 4-manifold of genus \(g\). Let \(m = \min(b_1(X), b_2(X))\).

1. If \(\xi^2 \equiv \sigma(X) \mod 16\), then either \(\xi^2 = \sigma(X)\) or, if not,
   a. If \(\xi^2 \equiv 0\) or \(\xi^2\) and \(\sigma(X)\) have the same sign, then \(|\xi^2 - \sigma(X)|/8 \leq m + g - 1\).
   b. If \(\sigma(X) = 0\) or \(\xi^2\) and \(\sigma(X)\) have opposite signs, then \(|\xi^2 - \sigma(X)|/8 \leq m + g - 2\).

2. If \(\xi^2 \equiv \sigma(X) + 8 \mod 16\), then
   a. If \(\xi^2 \equiv -8\) or \(\xi^2 + 8\) and \(\sigma(X)\) have the same sign, then \(|\xi^2 + 8 - \sigma(X)|/8 \leq m + g + 1\).
   b. If \(\sigma(X) = 0\) or \(\xi^2 + 8\) and \(\sigma(X)\) have opposite signs, then \(|\xi^2 + 8 - \sigma(X)|/8 \leq m + g\).

Theorem 2.2 (Gilmer [10] and Viro [25]). Let \(X\) be an oriented compact 4-manifold with \(\partial X = S^3\), and \(K\) a knot in \(\partial X\). Suppose that \(K\) bounds a surface of genus \(g\) in \(X\) representing an element \(\xi\) in \(H_2(X; \mathbb{Z})\).

1. If \(\xi\) is divisible by an odd prime \(d\), then \(|(d^2 - 1)/(2d^2)\xi^2 - \sigma(X) - \sigma_d(K)| \leq \dim H_2(X; \mathbb{Z}_d) + 2g\).
2. If \(\xi\) is divisible by 2, then \(|\xi^2/2 - \sigma(X) - \sigma(K)| \leq \dim H_2(X; \mathbb{Z}_2) + 2g\).

Theorem 2.3 (Robertello [20]). Let \(X\) be an oriented compact 4-manifold with \(\partial X = S^3\), and \(K\) a knot in \(\partial X\). Suppose that \(K\) bounds a disk in \(X\) representing a characteristic element \(\xi\) in \(H_2(X; \partial X)\). Then \((\xi^2 - \sigma(X))/8 \equiv \text{Arf}(K)\) (mod 2).

Lemma 2.1. If \(K\) is a knot obtained by a \((-1, \omega)\)-twisting from the unknot \(K_0\), then \(K\) bounds a properly embedded smooth disk \((D, \partial D) \subset (\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4))\) such that \([D] = \omega\gamma \in H_2(\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4); \mathbb{Z})\).

Recall, for convenience of the reader, a proof of Lemma 2.1. As shown in Figure 4, let \(D\) be a disk on which the \((-1, \omega)\)-twisting is performed. Note that the \((+1)\)-Dehn surgery on \(\partial D = C\) changes \(K_0\) to \(K\). Regard \(K_0\) and \(D\) as contained in the boundary of a four-dimensional 0-handle \(h^0\). Then attach a 2-handle \(h^2\) to \(h^0\) along \(\partial D\) with framing \(+1\). Since \(\mathbb{CP}^2 = h^0 \cup h^2 \cup h^3\) with \(h^0 \cong B^4\) and \(h^3 \cong B^4\). \(\square\)
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Figure 4

$\mathcal{C}_1$

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$\mathcal{C}_p$

$\mathcal{C}_q$

$\mathcal{C}'_1$

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Figure 5 The link $L(p, q)$

$B^4$, the resulting 4-manifold $h^0 \cup h^2$ is diffeomorphic to $\mathbb{CP}^2 - B^4$ (see [15]). Let $(\Delta, \partial \Delta) \subset (B^4, \partial B^4 \cong S^3)$ be a compact and orientable disk with $\partial \Delta = K_0$. Since $\text{lk}(K_0, \partial D) = \omega$, we can check that $[\Delta] = \omega \gamma \in H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$, where $\gamma$ is the standard generator of $H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$.

**Lemma 2.2 (Nakanishi [18]).** Suppose that $K$ is obtained from a trivial knot $K_0$ by $(n, \omega)$-twisting. If $\omega$ is even, then $e_2(K) \leq 2$.

**Lemma 2.3 (Ait Nouh [2]).** The $d$-signature of a $(2, q)$-torus knot $T(2, q)$ is given by the formula

$$\sigma_d(T(2, q)) = -(q - 1) - \left[\frac{q}{2d}\right].$$

To prove Theorem 1.1, we recall the definition of band surgery.

Let $L$ be a $c$-component oriented link. Let $B_1, \ldots, B_b$ be mutually disjoint oriented bands in $S^3$ such that $B_i \cap L = \partial B_i \cap L = \alpha_i \cup \alpha'_i$, where $\alpha_1, \alpha'_1, \ldots, \alpha_b, \alpha'_b$ are disjoint connected arcs. The closure of $L \cup \partial B_1 \cup \cdots \cup \partial B_b$ is also a link $L'$.

**Definition 2.1.** If $L'$ has the orientation compatible with the orientation of $L - \bigcup_{i=1}^b \alpha_i \cup \alpha'_i$ and $\bigcup_{i=1}^b (\partial B_i - \alpha_i \cup \alpha'_i)$, then $L'$ is called the link obtained by the band surgery along the bands $B_1, \ldots, B_b$. If $c = b + 1$, then this operation is called a fusion.

**Example 2.3.** Let $L(p, q) = C_1 \cup \cdots \cup C_p \cup C'_1 \cup \cdots \cup C'_q$ denote the $((p, 0), (q, 0))$-cable on the Hopf link with linking number 1 (see Figure 5). Then
Figure 6

$T(2,5)$ (resp. $T(2,7)$) can be obtained from $L(2,2)$ (resp. $L(2,4)$) by fusion (see Figure 6).

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following proposition.

Proposition 3.1. $T(2,p) \# T(2,q) \# T(2,r)$ is obtained from $L(2, g^* + \ell)$ by adding $b = g^* + \ell + 5$ bands, where $g^*$ denotes the 4-ball genus of $T(2,p) \# T(2,q) \# T(2,r)$, and $\ell$ is the number of integers in the set \{p, q, r\} that are congruent to 3 modulo 4. In particular, there is a cobordism of genus two between $L(2, g^* + \ell)$ and $T(2,p) \# T(2,q) \# T(2,r)$, where $g^* + \ell$ is always even.

Proof. Figure 7 shows that if $p \equiv 1 \pmod{4}$ (resp. $p \equiv 3 \pmod{4}$), then $T(2,p)$ is obtained from $L(2, \frac{p-1}{2})$ (resp. $L(2, \frac{p+1}{2})$) by fusion, that is, by adding $\frac{p-1}{2} + 1$ (resp. $\frac{p+1}{2} + 1$) bands. Therefore, to prove the proposition, there are four cases to distinguish:

Case I. $p \equiv q \equiv r \equiv 1 \pmod{4}$.

Case II. $p \equiv 3$ and $q \equiv r \equiv 1 \pmod{4}$. 
By a band surgery with \( b = 2 \), \( L(2, g^* + \ell) \) can be turned into a connected sum of \( L(2, \frac{p+1}{2}) \), \( L(2, \frac{q+1}{2}) \), \( L(2, \frac{r+1}{2}) \), which has \( g^* + \ell + 4 \) components.

Since each of the summands can be turned into \( T(2, p) \), \( T(2, q) \), \( T(2, r) \), respectively, by a fusion, we have that \( T(2, p) \# T(2, q) \# T(2, r) \) can be obtained from \( L(2, g^* + \ell) \) by a band surgery with \( b = g^* + \ell + 5 \).

Since each of the summands can be turned into \( T(2, p) \), \( T(2, q) \), \( T(2, r) \), respectively, by a fusion, we have that \( T(2, p) \# T(2, q) \# T(2, r) \) can be obtained from \( L(2, g^* + \ell) \) by a band surgery with \( b = g^* + \ell + 5 \).

Since the proofs of these cases are similar, we provide more details for the case \( \ell = 0 \).

**Case I.** \( p \equiv q \equiv r \equiv 1 \) (mod 4).

This is equivalent to \( \ell = 0 \). As shown in Figures 7 and 8, \( k = T(2, p) \# T(2, q) \# T(2, r) \) can be obtained from the link \( L(2, \frac{p+1}{2} + \frac{q+1}{2} + \frac{r+1}{2}) = L(2, g^*) \) by adding the number of bands equal to

\[
b = \frac{p - 1}{2} + \frac{q - 1}{2} + \frac{r - 1}{2} + 5
\]

\[
= g^* + 5.
\]

Note that \( c = \frac{p-1}{2} + \frac{q-1}{2} + \frac{r-1}{2} + 2 \) or, equivalently, \( c = g^* + 2 \). Since \( g_c = \frac{1-c+b}{2} \), we have that \( g_c = 2 \) and \( g^* + \ell = g^* \) is even.

Note that in all four cases, \( b = g^* + \ell + 5 \) and \( c = g^* + \ell + 2 \), and, therefore, there is a cobordism of genus \( g_c = \frac{1+c-b}{2} = 2 \) (see [9]) between \( L(2, g^* + 3) \) and \( k \).

**Proof of Theorem 1.1.** Assume for a contradiction that \( K \cong T(2, p) \# T(2, q) \# T(2, r) \) can be obtained by \((n, \omega)\)-twisting from an unknot \( K_0 \). Since \( e_2(T(2, p) \# T(2, q) \# T(2, r)) > 2 \), by Lemma 2.2, \( \omega \) is odd. Since \( K \) is a composite knot, \( n = \pm 1 \) (see [13; 12]). The following proofs are based on the gluing of two punctured standard 4-manifolds, as depicted in Figure 9.

**Case I.** Assume that \( n = +1 \). Then \( \tilde{K} = T(-2, p) \# T(-2, q) \# T(-2, r) \) can be obtained by \((1, \omega)\)-twisting along an unknot \( \tilde{K}_0 \), the inverse of the mirror-image of \( K_0 \) (see [3]). By Lemma 2.1 this yields that \( \tilde{K} \) bounds a disk \( (D, \partial D) \subset (\mathbb{C}P^2 - B^2, \partial (\mathbb{C}P^2 - B^2) \cong S^3) \) such that \( [D] = \omega \gamma \in H_2(\mathbb{C}P^2 - B^2, S^3; \mathbb{Z}) \), where \( \gamma \) denotes the standard generator of \( H_2(\mathbb{C}P^2; \mathbb{Z}) \) with \( \gamma^2 = +1 \).

On the other hand, there exist a 4-ball \( J \) and a mutually disjoint union of \( g^* + \ell + 2 \) properly embedded 2-disks \( \Delta_1, \Delta_2, \ldots, \Delta_{g^*+\ell+2} \) such that \( \Delta = \Delta_1 \) for a contradiction. Then \( K \cong T(2, p) \# T(2, q) \# T(2, r) \) can be obtained from \( L(2, g^* + \ell) \).
Case I: $p \equiv q \equiv r \equiv 1 \pmod{4}$

Since $K$ is obtained from $L(2, g^* + \ell)$ by the band surgery described in Proposition 3.1, there exists a $(g^* + \ell + 3)$-punctured genus-two surface $\tilde{F}$ in $S^3 \times [0, 1] \subset J$ such that we can identify this band surgery with $\tilde{F} \cap (S^3 \times (1/2))$, $\partial \tilde{F} = L(2, g^* + \ell) \cup k$ with $L(2, g^* + \ell)$ lies in $S^3 \times [0] \cong \partial J \times [0]$, and $K$ lies in $S^3 \times \{1\} \cong \partial J \times \{1\}$. The 3-sphere $S^3 \times \{1\}$ bounds a 4-ball $B^4 \subset J$. The surface $F = \Delta \cup \tilde{F}$ is a smooth genus-two surface properly embedded in $S^2 \times S^2 - B^4$ and with boundary $K$ such that

$$[F] = 2\alpha + (g^* + \ell)\beta \in H_2(S^2 \times S^2 - B^4, \partial(S^2 \times S^2 - B^4) \cong S^3; \mathbb{Z}).$$
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The genus-two smooth and closed surface $\Sigma = F \cup D$ satisfies

$$[\Sigma] = 2\alpha + (g^* + \ell)\beta + \omega\gamma \in H_2(S^2 \times S^2 \# \mathbb{CP}^2; \mathbb{Z}).$$

By Lemma 2.2, $\omega$ is odd, and by Proposition 3.1, $g^* + \ell$ is even. Then, $\xi = [\Sigma]$ is a characteristic class in $H_2(S^2 \times S^2 \# \mathbb{CP}^2; \mathbb{Z})$. Furthermore, $X = S^2 \times S^2 \# \mathbb{CP}^2$ is homeomorphic to $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ (e.g., see Scorpan’s book [21], p. 239, or Corollary 4.3 in Kirby’s book [15], p. 11). Note that $\xi^2$ and $\sigma(X)$ have the same signs, $m = 1$, and $g = 2$. Therefore, by Theorem 2.1(1)(a) and Theorem 2.1(2)(a),

$$\frac{|\xi^2 - \sigma(X)|}{8} \leq 3,$$

or, equivalently,

$$\frac{4(g^* + \ell) + \omega^2 - 1}{8} \leq 3.$$

This yields that the only possibilities are $g^* = 3$ or $4$ and $\omega = \pm 1$; equivalently, $K = T(2, 3) \# T(2, 3) \# T(2, 3)$, then $\ell = 3$ or $K = T(2, 3) \# T(2, 3) \# T(2, 5)$, and then $\ell = 2$ with $\omega = \pm 1$. Then $K$ would bound a disk $(D, \partial D) \subset (\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4))$ such that

$$\xi = [D] = \pm \tilde{\gamma} \in H_2(\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4); \mathbb{Z}),$$

where $\tilde{\gamma}$ is the standard generator of $H_2(\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4); \mathbb{Z})$ with $\tilde{\gamma}^2 = -1$, and therefore $|\xi^2 - \sigma(X)|/8 = 0$. This contradicts Theorem 2.3 since $\text{Arf}(K) = 1$. 

\[\text{Figure 9}\]
Case II. Assume that \( n = -1 \). Then there are two cases to exclude.

Case II(a). If \( \omega \) is divisible by a prime \( d \geq 3 \), then by Lemma 2.1, \( k \) bounds a smooth disk \( (D, \partial D) \subset (\mathbb{C}P^2 - B^4; S^3; \mathbb{Z}) \) such that

\[
\xi = [D] = \omega \gamma \in H_2(\mathbb{C}P^2 - B^4; S^3; \mathbb{Z})
\]

By Lemma 2.3 the signatures are

\[
\sigma(K) = -(p + q + r - 3) \quad \text{and} \quad \sigma_d(K) = -(p - 1) - \left( \frac{p}{2d} \right) - (q - 1) - \left( \frac{q}{2d} \right) - (r - 1) - \left( \frac{r}{2d} \right) \quad \text{(see [2])}
\]

This contradicts Theorem 2.2.

Case II(b). If \( \omega = \pm 1 \), then by the same argument as in Case I, this would yield the existence of a genus-two surface that satisfies

\[
\xi = [\Sigma] = 2\alpha + (g^* + \ell)\beta + \bar{\gamma} \in H_2(S^2 \times S^2 \# \mathbb{C}P^2; \mathbb{Z})
\]

If we let \( X = S^2 \times S^2 \# \mathbb{C}P^2 \), then \( \xi^2 \) and \( \sigma(X) \) have opposite signs with \( m = 1 \) and \( g = 2 \). Therefore, by Theorem 2.1(1)(b) and Theorem 2.1(2)(b),

\[
\frac{\xi^2 - \sigma(X)}{8} \leq 2
\]

or, equivalently, \( g^* + \ell \leq 4 \). This yields that the only possibilities are \( g^* = 3 \) or 4; equivalently, \( K = T(2, 3) \# T(2, 3) \# T(2, 3) \), then \( \ell = 3 \) or \( K = T(2, 3) \# T(2, 3) \# T(2, 5) \), and then \( \ell = 2 \). Therefore, \( g^* + \ell = 6 \), a contradiction. \( \square \)

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References


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Department of Mathematical Sciences, Bell Hall 144
The University of Texas at El Paso
500 University Avenue
El Paso, TX 79968
USA
manouh@utep.edu
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