Classification of surface-knots with four triple points

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Abstract. We classify all surface-knots with minimal number of triple points equals to four. We show that there are always two positive and two negative triple points. We will show that there are 62 possibilities.

Theorem 1. *Any surface-knot with four triple points must have two positive and two negative triple points with. Furthermore, the number of isotopy classes is* 62*.*

Proof. S. Satoh found in [1] the following obstruction on the projection of a surface-knot F in $\mathbb{R}^3$. For any $\lambda \in \mathbb{Z}$, we have:

$$t_0(\lambda) + 2t_2(\lambda) + t_5(\lambda) + 2t_{25}(\lambda) = t_0(\lambda+1) + t_2(\lambda+1) + 2t_5(\lambda+1) + 2t_{25}(\lambda+1)$$

Where $\omega \in \{0, 2, 5, 25\}$ is the type of a triple point T. We will classify the surface-knots with four triple points T_1, T_2, T_3, T_4 according to the Alexander numbering and their types (see S. Satoh [2] for definitions). Let $\epsilon(T_i)$ be the sign of T_i for $i = 1, 2, 3, 4$.

Case 1. All triple points have the same Alexander numbering, say λ.

Case 1.1. $\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega$.

By Satoh's identity, the only possibity is that we have two positive and two negative triple points with the possibility that any type can occur. Therefore, we have four possibilities. Note that in case $\omega =< 25 >$, the genus of F is higher than 0 (see Hasegawa [3]).

Case 1.2. $\omega_i \neq \omega_j$ for any $i, j \in \{1, 2, 3, 4\}$

By Satoh's identity, the only possibity is that we have two positive and two negative triple points in which cases both T_1 and T_3 are positive with types $< 2 >$ and $< 0 >$ and T_2 and T_4 are negative with types $< 25 >$ and $< 5 >$ respectively or vice-versa. Therefore there are four cases.

Case 1.3. $\omega_1 = \omega_2 \neq \omega_3 \neq \omega_4$.

By Satoh's identity, there are two possibilities in both $\epsilon(T_1) = \epsilon(T_3) = +1$ with $\epsilon(T_2) = \epsilon(T_4) = -1$

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Possibility I. $\omega_1 = \omega_2 =< 0 >$, $\omega_3 =< 2 >$, $\omega_4 =< 25 >$

Possibility II. $\omega_1 = \omega_2 =< 0 >$, $\omega_3 =< 2 >$, $\omega_4 =< 25 >$

Case 1.4. $\omega_1 = \omega_2 = \omega_3 \neq \omega_4$.

By Satoh's identity, we have four possibities in all of them $\epsilon(T_1) = \epsilon(T_2) = +1$ and $\epsilon(T_3) = \epsilon(T_4) = -1$.

Possibility I: $\omega_1 = \omega_2 = \omega_3 =< 0 >$ and $\omega_4 =< 5 >$.

Possibility II: $\omega_1 = \omega_2 = \omega_3 =< 5 >$ and $\omega_4 =< 0 >$.

Possibility III: $\omega_1 = \omega_2 = \omega_3 =< 2 >$ and $\omega_4 =< 25 >$.

Possibility IV: $\omega_1 = \omega_2 = \omega_3 =< 25 >$ and $\omega_4 =< 2 >$.

Case 2. The Alexander numbering of the triple points are not all equal.

Claim. $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$

Proof. We can assume that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ without loss of generality. Assume for a contradiction that $\lambda_1 < \lambda_2$, and let $\lambda = \lambda_1 - 1$ in Satoh's identity. Therefore, we have

$$0 = t_0(\lambda_1) + t_2(\lambda_1) + 2t_5(\lambda_1) + 2t_{25}(\lambda_1)$$

This right-hand equation is equal to $\pm\epsilon(T_1)$ or $\pm 2\epsilon(T_1)$, a contradiction. Therefore, $\lambda_1 = \lambda_2$. Assume now for a contradiction that $\lambda_3 < \lambda_4$, and let $\lambda = \lambda_4$ in Satoh's identity. Therefore, we have

$$t_0(\lambda_4) + 2t_2(\lambda_4) + t_5(\lambda_4) + 2t_{25}(\lambda_4) = 0$$

This left-hand equation is equal to $\pm\epsilon(T_4)$ or $\pm 2\epsilon(T_4)$, a contradiction. Therefore, $\lambda_3 = \lambda_4$.

Case 2.1. If $\lambda_3 > \lambda_2 + 1 (= \lambda_1 + 1)$, then let $\lambda = \lambda_2$ in Satoh's identity. This implies that

$$t_0(\lambda_2) + 2t_2(\lambda_2) + t_5(\lambda_2) + 2t_{25}(\lambda_2) = 0$$

Let Possibility I and Possibility II be the main possibilities.

Possibility I: $\epsilon(T_1) = \epsilon(T_3) = +1$ and $\epsilon(T_2) = \epsilon(T_4) = -1$ with types $\omega_1 = \omega_3 =< 0 >, \omega_2 =< 5 >$ and $\omega_4 =< 2 >$.

Possibility II: $\epsilon(T_1) = \epsilon(T_3) = +1$ and $\epsilon(T_2) = \epsilon(T_4) = -1$. $\omega_1 =< 2 >, \omega_2 =< 25 >, \omega_3 =< 5 >$ and $\omega_4 =< 25 >$.

Note that we can intechange T_1 and T_2 (resp. T_3 and T_4) and their types. There are sixteen such possibilities.

Case 2.1. If $\lambda_3 = \lambda_2 + 1$. By letting $\lambda = \lambda_2$ and $\lambda = \lambda_4$ in Satoh's identity, we obtain

(1) $t_0(\lambda_1) + 2t_2(\lambda_1) + t_5(\lambda_1) + 2t_{25}(\lambda_1) = 0$

(2) $t_0(\lambda_2)+2t_2(\lambda_2)+t_5(\lambda_2)+2t_{25}(\lambda_2) = t_0(\lambda_2+1)+t_2(\lambda_2+1)+2t_5(\lambda_2+1)+2t_{25}(\lambda_2+1)$

Case 2.1.1**.** If $\omega_1 = \omega_2$ and $\omega_3 = \omega_4$ then we must have $\epsilon(T_1) = \epsilon(T_3) = +1$ and $\epsilon(T_2) = \epsilon(T_4) = -1$. Therefore, there are sixsteen possibilities. Note that the case where $\omega_1 = \omega_2 =< 0 >$ and $\omega_3 = \omega_4 =< 25 >$ correspond to the case where the genus of F would be greater than zero, which contradict that F is a 2-knot (see I. Hasegawa [3]).

Case 2.1.2**.** If $\omega_i \neq \omega_j$ for any $i, j \in \{1, 2, 3, 4\}$, then we must have $\epsilon(T_1) = \epsilon(T_3) = \epsilon$ and $\epsilon(T_2) = \epsilon(T_4) = -\epsilon$. We have sixsteen possibilities given by the types. By Hasegawa's theorem, the case $\omega_1 = \omega_3 =< 0 >$ and $\omega_2 = \omega_4 =< 25 >$ corresponds to the case where the genus of F is higher than 0.

References

[1] S. Satoh and A. Shima, Positive, alternating, and pseudo-ribbon surface-knots, Kobe J. Math. **19** (2002), 51-59.

[2] S. Satoh, No 2-knot has triple point number two or three, preprint.

[3] I. Hasegawa, The minimum ω-index of non-ribbon surface-links, preprint