Mountain pass solutions to equations of p-Laplacian type

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Abstract

This work is devoted to study the existence of solutions to equations of the p-Laplacian type. We prove the existence of at least one solution, and under further assumptions, the existence of infinitely many solutions. In order to apply mountain pass results, we introduce a notion of uniformly convex functional that generalizes the notion of uniformly convex norm.

Keywords: p-Laplacian, multiple solutions, Mountain pass Theroem, uniform convexity, Clarkson inequality.

1 Introduction

In the literature the mountain pass theorem ([3]) has been applied to find solutions to quasilinear elliptic equations of p-Laplacian type. More specifically, in [9] Dinca, Jebelean and Mawhin used it to solve the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u &= f(x,u) & \text{in} & \Omega \\ u &= 0 & \text{on} & \partial\Omega \end{cases}$$

with $\Omega \subset \mathbb{R}^n$ a bounded domain, $n \geq 2$ and f satisfying natural growth conditions (see also [10]). In [4] existence results are obtained for more general quasilinear equations in the framework of Orlicz-Sobolev spaces. In [2] another minimax theorem: Rabinowitz' saddle point theorem, is applied to prove a Landesman-Lazer type result for a problem involving a p-Laplacian. For other related results, see for example [7], [8], [11], [12].

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In this work, we obtain existence and multiplicity results for equations involving more general elliptic operators in divergence form:

$$-\mathrm{div}(a(x,\nabla u)) = f(x,u) \tag{1}$$

Such operators arise, for example, from the expression of the p-Laplacian in curvilinear coordinates. In order to apply the mountain pass theorem to this kind of equations, it is necessary to check that the associated functional verifies the Palais-Smale condition. In the case of the p-Laplacian, this is usually achieved by using the uniform convexity of the Sobolev space $W_0^{1,p}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,p}(\Omega} = \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p} \tag{2}$$

In order to extend this idea to more general equations, we introduce a notion of uniformly convex functional. Under suitable assumptions, we prove the existence of at least one solution to equation (1) with Dirichlet condition (theorem 3.1). Finally, for f and a odd in their second argument, we prove the existence of infinitely many solutions (theorem 4.1).

2 Uniformly Convex Functionals

In this section, X denotes a Banach space.

Definition 2.1 *We shall say that the convex functional $A : X \to \mathbb{R}$ is uniformly convex on the (convex) set $C \subset X$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$A\left(\frac{x+y}{2}\right) \leq \frac{1}{2}A(x) + \frac{1}{2}A(y) - \delta(\varepsilon)$$

for $x, y \in C$ and $\|x-y\| > \varepsilon$. If A is uniformly convex on every ball of X, we shall say that A is locally uniformly convex.

Remark 2.1 *X is uniformly convex if and only if its norm is locally uniformly convex.*

The uniform convexity of $W_0^{1,p}(\Omega)$ with the norm (2) is proved for $p \geq 2$, using the first Clarkson inequality (see [1], theorem 2.28):

$$\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \leq \frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p \; \forall \, u, v \in L^p(\Omega)$$

(For $1 < p < 2$, $W_0^{1,p}(\Omega)$ is still uniformly convex, but a different inequality is used for proving that.) In order to give examples of uniformly convex functionals, we give some conditions for a functional to satisfy an analogous inequality.

Lemma 2.1 *Let $F : \mathbb{R} \to \mathbb{R}$ be a C^1 function and $1 < p < \infty$. Assume that F' is strongly p-monotone in the following sense:*

$$\frac{F'(y) - F'(x)}{(y-x)^{p-1}} \geq c \; \forall x, y \in \mathbb{R} \text{ such that } y > x$$

then F verifies the following Clarkson type inequality:

$$F\left(\frac{x+y}{2}\right) \leq \frac{1}{2}F(x) + \frac{1}{2}F(y) - \frac{c}{2^{p+1}}|x-y|^p \tag{3}$$

Proof: Let $z = \frac{x+y}{2}$. We have

$$F(y) - F(z) = \int_z^y F'(t)dt$$

$$F(z) - F(x) = \int_x^z F'(s)ds$$

Subtracting and setting $t = s + \frac{y-x}{2}$, we conclude that:

$$\frac{1}{2}F(x) + \frac{1}{2}F(y) - F(z) = \frac{1}{2}\int_x^z \left[F'\left(s + \frac{y-x}{2}\right) - F'(s)\right] ds$$

On the other hand,

$$F'\left(s + \frac{y-x}{2}\right) - F'(s) \geq c\left(\frac{y-x}{2}\right)^{p-1}$$

We deduce that:

$$\frac{1}{2}F(x) + \frac{1}{2}F(y) - F(z) \geq \frac{c}{2}\left(\frac{y-x}{2}\right)^p$$

and the proof is complete.

Example: $x \mapsto x^p$ is strongly p-monotone if $p \geq 2$.

Definition 2.2 *Let $F : \mathbb{R} \to \mathbb{R}$ be a C^1 function. If F verifies an inequality of the form (3) for some constant $c > 0$, we shall say that it is* p**-uniformly convex.**

Corollary 2.1 *Assume that $F : \mathbb{R} \to \mathbb{R}$ verifies that $|F(x)| \leq c(1 + |x|^p)$, and that it is strongly p-monotone. For $u \in L^p(\Omega)$ let us define $A(u) = \int_\Omega F(u)$, then A is a locally uniformly convex functional.*

The following lemma generalizes the first Clarkson inequality.

Lemma 2.2 *Assume that A is a p-uniformly convex function and that $\alpha > 1$, then $A(x)^\alpha$ is q-uniformly convex with $q = \alpha p$.*

Proof: By assumption A is p-uniformly convex, then there exists a constant $c > 0$ such that:

$$A\left(\frac{x+y}{2}\right) + c\left|\frac{x-y}{2}\right|^p \leq \frac{1}{2}A(x) + \frac{1}{2}A(y)$$

Since $\alpha > 1$, we have

$$s^\alpha + t^\alpha \leq (s+t)^\alpha \; \forall s,t \geq 0$$

Choosing $s = A\left(\frac{x+y}{2}\right)$ and $t = c\left|\frac{x-y}{2}\right|^p$ we have:

$$A\left(\frac{x+y}{2}\right)^\alpha + c^\alpha\left|\frac{x-y}{2}\right|^q \leq \left[A\left(\frac{x+y}{2}\right) + c\left|\frac{x-y}{2}\right|^p\right]^\alpha$$

Note that $x \mapsto x^\alpha$ is increasing, therefore we deduce that:

$$A\left(\frac{x+y}{2}\right)^\alpha + c^\alpha\left|\frac{x-y}{2}\right|^q \leq \left[\frac{1}{2}A(x) + \frac{1}{2}A(y)\right]^\alpha$$

Now as $x \mapsto x^\alpha$ is convex, we have that:

$$A\left(\frac{x+y}{2}\right)^\alpha + c^\alpha\left|\frac{x-y}{2}\right|^q \leq \frac{1}{2}A(x)^\alpha + \frac{1}{2}A(y)^\alpha$$

This means that A^α is q-uniformly convex.

Examples: 1) If we take $A(x) = |x|^2$ we see that $|x|^p$ is p-uniformly convex if $p \geq 2$, and then we obtain the usual Clarkson inequality. In fact, we have:

$$\left|\frac{x+y}{2}\right|^p + \left|\frac{x-y}{2}\right|^p \leq \frac{|x|^p}{2} + \frac{|y|^p}{2}$$

2) If $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $Q(x) = \langle Ax, x\rangle$ is the associated quadratic form, then Q is 2-uniformly convex. In fact, using the polarization formula:

$$Q\left(\frac{x+y}{2}\right) + Q\left(\frac{x-y}{2}\right) = \frac{1}{2}Q(x) + \frac{1}{2}Q(y)$$

A is definite positive, then:

$$Q\left(\frac{x-y}{2}\right) \geq c|x-y|^2$$

It follows that

$$Q\left(\frac{x+y}{2}\right) \leq \frac{1}{2}Q(x) + \frac{1}{2}Q(y) - c|x-y|^2$$

We conclude that Q is 2-uniformly convex. By the previous lemma $Q(x)^{p/2}$ is p-uniformly convex when $p \geq 2$. A more general way to get uniformly convex functionals is given by the following lemma:

Lemma 2.3 *Let $A : X \to \mathbb{R}$ be a functional of class C^2. Assume that D^2A is uniformly positive definite, i.e.: there exists a constant $c > 0$ such that*

$$D^2A(x)(\xi, \xi) \geq c|\xi|^2 \; \forall \, x, \xi \in X$$

then A is 2-uniformly convex.

Proof: For $x, y \in X$, we define $g : [0, 1] \to \mathbb{R}$, $g(t) = A((1-t)x + ty)$. Then:

$$g'(t) = DA((1-t)x + ty)(y - x)$$

$$g''(t) = D^2A((1-t)x + ty)(y - x, y - x) \geq c|x - y|^2$$

by hypothesis. From the mean value theorem, for $s, t \in [0, 1]$

$$\frac{g'(s) - g'(t)}{s - t} = g''(\theta) \geq c|x - y|^2$$

(for some $\theta \in [0, 1]$). By lemma (2.1) we deduce that:

$$g\left(\frac{s+t}{2}\right) \leq g(s) + g(t) - \frac{c}{4}|x - y|^2|s - t|^2$$

Choosing $s = 1$ and $t = 0$, we obtain:

$$A\left(\frac{x+y}{2}\right) \leq \frac{1}{2}A(x) + \frac{1}{2}A(y) - \frac{c}{4}|x - y|^2$$

We conclude that A is 2-uniformly convex.

2.1 The (S_+) condition

Definition 2.3 *An operator $a : X \to X^*$ verifies the (S_+) condition if for any sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \rightharpoonup x$ and*

$$\limsup_{n \to \infty} \langle a(x_n), x_n - x \rangle \leq 0$$

we have that $x_n \to x$ strongly.

Proposition 2.1 *Suppose that $A : X \to \mathbb{R}$ is a C^1 locally uniformly convex functional that is locally bounded. Then $a = DA : X \to X^*$ verifies the (S_+) condition.*

Proof: Let (x_n) be a sequence in X such that $x_n \rightharpoonup x$ and

$$\limsup_{n \to \infty} \langle a(x_n), x_n - x \rangle \leq 0$$

As (x_n) is weakly convergent, it is bounded: $\|x_n\| < R$. Since A is locally bounded, we deduce that $A(x_n)$ is bounded. For a subsequence, we may assume that $A(x_n) \to c$. By weakly lower semicontinuity:

$$A(x) \leq \liminf A(x_n) = c$$

On the other hand, since A is convex, its graphic lies above the tangent hyperplane at x_n:

$$A(x) \geq A(x_n) + \langle a(x_n), x - x_n \rangle$$

Using that $\limsup_{n\to\infty} \langle a(x_n), x_n - x \rangle \leq 0$, we deduce that $A(x) \geq c$, then, $A(x) = c$.
Also we have that $\frac{x_n + x}{2} \rightharpoonup x$, and again by lower weakly semicontinuity:

$$c = A(x) \leq \liminf A\left(\frac{x + x_n}{2}\right) \tag{4}$$

If we suppose that (x_n) does not converge strongly to x, then there exists $\varepsilon > 0$ and a subsequence (x_{n_k}) that verifies $\|x - x_{n_k}\| \geq \varepsilon$. Let $\delta > 0$ corresponding to ε by uniform convexity of A over the ball $B(0, R)$. Then:

$$\frac{1}{2}A(x) + \frac{1}{2}A(x_{n_k}) - A\left(\frac{x + x_{n_k}}{2}\right) \geq \delta(\varepsilon)$$

As $k \to \infty$, we have that:

$$\limsup A\left(\frac{x + x_{n_k}}{2}\right) \leq c - \delta(\varepsilon)$$

which contradicts (4).

3 Applications to p-Laplacian type equations

Theorem 3.1 *Let us assume that the following conditions hold:*

1. $p > 1$ *and* $\Omega \subset \mathbb{R}^N$ *is a bounded domain.*
2. *Let* $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x, \xi)$ *be a continuous function in* $\overline{\Omega} \times \mathbb{R}^N$, *with continuous derivative with respect to* ξ, $a = DA = A'$, *and suppose that the following conditions hold:*
   (a) $A(x, 0) = 0 \; \forall \; x \in \Omega$.
   (b) *a satisfies the growth condition:*
   $$|a(x, \xi)| \leq c_1(1 + |\xi|^{p-1}) \; \forall \, x \in \Omega, \; \xi \in \mathbb{R}^N \tag{5}$$
   for some constant $c_1 > 0$.
   (c) *A is p-uniformly convex: There exists a constant* $k > 0$ *such that:*
   $$A\left(x, \frac{\xi + \eta}{2}\right) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \eta) - k|\xi - \eta|^p \; \forall \, x \in \overline{\Omega}, \; \xi, \eta \in \mathbb{R}^N$$
   (d) *A is p-subhomogeneous:*
   $$0 \leq a(x, \xi) \cdot \xi \leq pA(x, \xi) \;\; \forall \, x \in \overline{\Omega}, \; \xi \in \mathbb{R}^N \tag{6}$$

(e) *A satisfies the ellipticity condition:*

$$A(x,\xi) \geq \Lambda|\xi|^p \; \forall \;\; x \in \overline{\Omega}, \xi \in \mathbb{R}^N \tag{7}$$

with $\Lambda > 0$ *a constant.*

3. $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function that verifies:*

   (a) *the subcritical growth condition:*

   $$|f(x,s)| \leq c(1 + |s|^{q-1}) \; \forall \, x \in \Omega, s \in \mathbb{R} \tag{8}$$

   where $p < q < p^* = \frac{Np}{N-p}$ *if* $p < N$*, or* $p < q < +\infty$ *if* $p \geq N$*.*

   (b) *the Ambrosetti-Rabinowitz condition:* $F(x,s) = \int_0^s f(x,t) \; dt$ *is* θ*-superhomogeneous at infinity, i.e.: There exists* $s_0 > 0$ *such that*

   $$0 < \theta F(x,s) \leq f(x,s)s \; \forall s \in \mathbb{R} \text{ such that } |s| \geq s_0, x \in \overline{\Omega} \tag{9}$$

   where $\theta > p$*.*

4. *We assume that*

   $$\limsup_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} \leq \lambda \tag{10}$$

   uniformly for $x \in \Omega$*, where*

   $$\lambda < \lambda_{1,p}(\Omega) p \Lambda$$

Then the Dirichlet problem:

$$\left\{ \begin{array}{lcll} -\mathrm{div}(a(x,\nabla u)) & = & f(x,u) & in \;\; \Omega \\ u & = & 0 & in \;\; \partial\Omega \end{array} \right. \tag{11}$$

has at least one nontrivial weak solution in $W_0^{1,p}(\Omega)$*.*

Some Examples:

1. $A(x,s) = \frac{1}{p}|s|^p$, $a(x,s) = |s|^{p-2}s$ with $p \geq 2$, and we get the usual p-Laplacian.

2. Let $(a^{ij}(x)) \in C(\overline{\Omega}, \mathbb{R}^{N\times N})$ be a symmetric positive definite matrix:

   $$a^{ij}(x) = a^{ji}(x)$$

   $$\sum_{i,j=1}^{N} a^{ij}(x) \; \xi_i \xi_j \geq c|\xi|^2 \; \forall \, x \in \overline{\Omega}, \xi \in \mathbb{R}^N \text{ and } c > 0$$

   then, setting

   $$A(x,\xi) = \frac{1}{2} \sum_{i,j=1}^{N} a^{ij}(x) \; \xi_i \xi_j$$

we obtain that the previous theorem can be applied to the linear elliptic operator in divergence form:

$$L[u] = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left(a^{ij}(x) \frac{\partial u}{\partial x_i} \right)$$

Remark 3.1 *These two cases have already been considered in the literature, but our result gives a unified approach to them, and other cases like the following.*

3. Let $p \geq 2$. If we consider $A(x,s) = \frac{1}{p}\left[(1+|s|^2)^{p/2} - 1\right]$ we get the generalized mean curvature operator

$$\mathrm{div}((1+|\nabla \mathrm{u}|^2)^{(\mathrm{p}-2)/2} \nabla \mathrm{u})$$

To see that A is p-uniformly convex, note that the function $1+|s|^2$ is 2-uniformly convex, so by proposition 2.2, the function $(1+|s|^2)^{p/2}$ is p-uniformly convex, and we conclude that A is also p-uniformly convex. Hence our result can be applied to this operator. It generalizes the prescribed mean curvature operator for a nonparametric surface, which corresponds to the case $p = 1$ ([5], [6]).

3.1 Some remarks on our assumptions

Remark 3.2 *Integrating (5) we deduce that A verifies the growth condition:*

$$|A(x,\xi)| \leq c_2(|\xi| + |\xi|^p) \;\; \forall\, x \in \overline{\Omega}, \xi \in \mathbb{R}^N \tag{12}$$

for some constant $c_2 > 0$.

In fact, we have:

$$A(x,\xi) = \int_0^1 \frac{d}{dt} A(x,t\xi)\, dt = \int_0^1 a(x,t\xi)\cdot \xi \, dt$$

From (5), we deduce that:

$$A(x,\xi) \leq c_1 \int_0^1 (1 + |\xi|^{p-1} t^{p-1})|\xi| dt \leq c_1|\xi| + \frac{c_1}{p}|\xi|^p$$

Remark 3.3 *Condition (6) implies that*

$$A(x,s\xi) \leq A(x,\xi) s^p \;\forall s \geq 1, x \in \overline{\Omega}, \xi \in \mathbb{R}^N$$

To see that, let us define $g(t) = A(x,t\xi)$, then

$$g'(t) = a(x,t\xi)\cdot\xi = \frac{1}{t} a(x,t\xi)\cdot(t\xi) \leq \frac{p}{t} A(x,t\xi) = \frac{p}{t} g(t)$$

$$\frac{g'(t)}{g(t)} \leq \frac{p}{t}$$

Integrating, we have:

$$\log\, g(s) - \log\, g(1) \leq p \log s$$

then

$$\frac{g(s)}{g(1)} \leq s^p$$

and we obtain that:

$$A(x, s\,\xi) \leq A(x,\xi)s^p$$

In a similar way, we have:

Remark 3.4 *Condition (9) implies that:*

$$F(x, sv) \geq F(x,v)s^\theta\, \forall v \text{ such that } |v| \geq s_0,\ s \geq 1, x \in \overline{\Omega}$$

3.2 Proof of theorem 3.1

Under the assumptions of theorem 3.1, we define the functional:

$$J(u) = \int_\Omega A(x, \nabla u) - \int_\Omega F(x,u)$$

$J : W_0^{1,p}(\Omega) \to \mathbb{R}$ is well defined and C^1. Its derivative is given by:

$$\langle J'(u), \varphi\rangle = \int_\Omega a(x,\nabla u)\cdot\nabla\varphi - \int_\Omega f(x,u)\varphi$$

In order to prove theorem 3.1, we apply to this functional the mountain pass theorem. In the next two lemmas, we shall verify the mountain pass theorem conditions

Lemma 3.1 *J verifies the Palais-Smale condition.*

Proof: Let $(u_n) \subset W_0^{1,p}(\Omega)$ be a Palais-Smale sequence, i.e: $J(u_n) \to c$ and $J'(u_n) \to 0$. We first show that (u_n) is bounded:

$$J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle = \int_\Omega \left[A(x,\nabla u_n) - \frac{1}{\theta}a(x,\nabla u_n)\cdot\nabla u_n\right] +$$

$$\int_\Omega \left[\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right]$$

From condition (6), we see that:

$$J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle \geq \left(1 - \frac{p}{\theta}\right)\int_\Omega A(x,\nabla u_n) +$$

$$\int_\Omega \left[\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right]$$

then

$$\left(1-\frac{p}{\theta}\right)\int_\Omega A(x,\nabla u_n) \le J(u_n) - \frac{1}{\theta}\left\langle J'(u_n),u_n\right\rangle$$
$$-\int_{\{x\in\Omega:|u_n(x)|>s_0\}}\left[\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right] + M|\Omega|$$

where $M=\sup\{\left|\frac{1}{\theta}f(x,s)s-F(x,s)\right| : x\in\overline{\Omega}, |s|\le s_0\}$. From the Ambrosetti-Rabinowitz condition (9), we deduce that:

$$\left(1-\frac{p}{\theta}\right)\int_\Omega A(x,\nabla u_n) \le J(u_n) - \frac{1}{\theta}\left\langle J'(u_n),u_n\right\rangle + M|\Omega|$$

From the ellipticity condition (7), we deduce that:

$$C\int_\Omega |\nabla u_n|^p \le J(u_n) - \theta\left\langle J'(u_n),u_n\right\rangle + M|\Omega|$$

with $C=\left(1-\frac{p}{\theta}\right)\Lambda>0$, or:

$$C\left\|u_n\right\|^p_{W_0^{1,p}(\Omega)} \le J(u_n) + \theta\left\|J'(u_n)\right\|\left\|u_n\right\|_{W_0^{1,p}(\Omega)} + M|\Omega|$$

If (u_n) is not bounded, we may assume, for a subsequence, that: $\|u_n\|_{W_0^{1,p}(\Omega)}\to\infty$. Then dividing by $\|u_n\|_{W_0^{1,p}(\Omega)}$ and letting $n\to\infty$, we obtain a contradiction. We conclude that (u_n) is bounded.
From the reflexivity of $W_0^{1,p}(\Omega)$ we may extract a weakly convergent subsequence, that for simplicity we call (u_n). So we may assume that $u_n\rightharpoonup u$. To prove that (u_n) converges strongly to u, we note that A is p-uniformly convex. Therefore, by proposition 2.1, it follows that if we define the functional:

$$J_0(u)=\int_\Omega A(x,\nabla u)$$

then J_0' verifies the (S_+) condition; so it is enough to prove that:

$$\limsup_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot(\nabla u_n-\nabla u)\le 0$$

We have that

$$\int_\Omega a(x,\nabla u_n)\cdot(\nabla u_n-\nabla u) = \left\langle J'(u_n),u_n-u\right\rangle + \int_\Omega f(x,u_n)\cdot(u_n-u)$$

Now, since the Sobolev embedding $W_0^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$, with q as in (8), is compact, we get (for a subsequence) that $u_n\to u$ strongly in $L^q(\Omega)$, and by the growth condition on f, $f(x,u_n)$ is bounded in $L^{q'}(\Omega)$. It follows that:

$$\int_\Omega f(x,u_n)\cdot(u_n-u)\to 0$$

On the other hand, since $J'(u_n) \to 0$ by assumption, and $u_n - u$ is bounded in $W_0^{1,p}(\Omega)$, we deduce that

$$\langle J'(u_n), u_n - u \rangle \to 0$$

We conclude that:

$$\int_\Omega a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) \to 0$$

and then from condition (S_+), $u_n \to u$ strongly.

Lemma 3.2 *J has the geometry of the mountain pass lemma, i.e.:*

1. *There exists $r > 0$ such that:*

$$\inf_{\|u\|=r} J(u) = b > 0$$

2. *There exists $u_0 \in W_0^{1,p}(\Omega)$ such that*

$$J(tu_0) \to -\infty \text{ as } t \to \infty$$

Proof: 1. We choose $\varepsilon > 0$ small enough, verifying:

$$\Lambda > \frac{\lambda + \varepsilon}{p\lambda_{1,p}}$$

then by condition (10), there exists $\delta > 0$ such that:

$$\frac{f(x,s)}{|s|^{p-2}s} \leq \lambda + \varepsilon \text{ for } |s| \leq \delta,\ \forall x \in \Omega$$

and we obtain that:

$$f(x,s) \begin{cases} \leq (\lambda+\varepsilon)s^{p-1} & \text{if} \quad s > 0 \\ \geq -(\lambda+\varepsilon)|s|^{p-1} & \text{if} \quad s < 0 \end{cases}$$

Integrating, we deduce that:

$$F(x,s) \leq \frac{1}{p}(\lambda+\varepsilon)|s|^p \text{ for } |s| \leq \delta$$

As a consequence, using hypothesis (8) and the Sobolev embedding for $q < p^*$ if $p < N$, or $q < \infty$ if $p \geq N$:

$$J(u) \geq \int_\Omega A(x, \nabla u) - \int_{\{x\in\Omega : |u| \leq \delta\}} \frac{(\lambda+\varepsilon)}{p}|u|^p - \int_{\{x\in\Omega : |u| > \delta\}} C|u|^q$$

$$\geq \Lambda \int_\Omega |\nabla u|^p - \frac{\lambda+\varepsilon}{p\lambda_{1,p}} \int_\Omega |\nabla u|^p - C\left(\int_\Omega |\nabla u|^p\right)^{q/p} = \phi(r)$$

with $r = \int_\Omega |\nabla u|^p$. Moreover $\phi(r) > 0$ for $r > 0$ small enough, since $q > p$.

2. Being A p-subhomogeneous, and $F(x,s)$ θ-superhomogeneous if $|s| \geq s_0$, for $t > 1$, we have that:

$$J(tu_0) = \int_\Omega A(x, t\nabla u_0) - \int_\Omega F(x, tu_0)$$

$$\leq t^p \int_\Omega A(x, \nabla u) - t^\theta \int_{\{x\in\Omega:|u_0|\geq s_0\}} F(x, u_0) + M|\Omega|$$

with $M = \sup\{|F(x,s)| : x \in \overline{\Omega}, |s| \leq s_0\}$. Since $\theta > p$, choosing u_0 such that

$$|\{x \in \Omega : u_0(x) \geq s_0\}| > 0$$

and using condition (9), we deduce that $J(tu_0) \to -\infty$ as $t \to +\infty$.

4 Multiple Solutions

In this section, we prove the existence of infinitely many solutions. Our main result is:

Theorem 4.1 *Assume that the Carathéodory functions $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are odd with respect to the second argument:*

$$f(x, -s) = -f(x, s) \text{ and } a(x, -s) = -a(x, s)$$

and that the conditions of theorem 3.1 are satisfied. Then problem (11) has infinitely many solutions.

We shall need the following "$\mathbb{Z}_2$-symmetric" version (for even functionals) of the mountain pass theorem (see [13], theorem 9.12).

Theorem 4.2 *Let X be an infinite dimensional real Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying the (PS) condition and $I(0) = 0$. Suppose that:*

- (I_1) *There exist constants $\rho, \alpha > 0$ such that $I(x) \geq \alpha$ if $\|x\| = \rho$.*
- (I_2) *For each finite dimensional subspace $X_1 \subset X$, the set $\{x \in X_1 : I(x) \geq 0\}$ is bounded.*

then I has an unbounded sequence of critical values.

In order to prove this result, we need the following lemma:

Lemma 4.1 *The conditions of the theorem imply that if $X_1 \subset W_0^{1,p}(\Omega)$ is a finite dimensional subspace, the set $S = \{u \in X_1 : J(u) \geq 0\}$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof: From condition (8), F satisfies:

$$|F(x,s)| \le C_1(|s|^q + 1)\, \forall x \in \Omega, s \in \mathbb{R} \tag{13}$$

We claim that there exists $\gamma \in L^\infty(\Omega)$, $\gamma > 0$ in Ω, such that:

$$F(x,s) \ge \gamma(x)|s|^\theta\, \forall x \in \Omega, |s| \ge s_0 \tag{14}$$

In fact, since F is θ-superhomogeneous, we have that:

$$F(x,s) \ge \gamma_1(x)|s|^\theta\, \forall x \in \Omega, s \ge s_0$$

where $\gamma_1(x) = \frac{F(x,s_0)}{s_0^\theta}$. By condition (13), $\gamma_1 \in L^\infty(\Omega)$ and from condition (9) we conclude that $\gamma_1 > 0$ in Ω. In a similar way, it follows that

$$F(x,s) \ge \gamma_2(x)|s|^\theta\, \forall x \in \Omega, s \le -s_0$$

with $\gamma_2 = \frac{F(x,-s_0)}{s_0^\theta}$. Again $\gamma_2 \in L^\infty(\Omega)$ and $\gamma_2 > 0$. Then (14) holds with $\gamma(x) = \min(\gamma_1(x), \gamma_2(x))$ for $x \in \Omega$, as we have claimed.
We shall prove that J satisfies an estimate of the form:

$$J(u) \le c_3\, \|u\|^p_{W_0^{1,p}(\Omega)} + c_4 \|u\|_{W_0^{1,p}(\Omega)} - \int_\Omega \gamma(x)|u|^\theta + K\, \forall u \in W_0^{1,p}(\Omega) \tag{15}$$

with c_3, c_4 and K non negative constants.
Let $v \in W_0^{1,p}(\Omega)$ arbitrary, and let us denote:

$$\begin{aligned} \Omega_< &= \{x \in \Omega : |v(x)| < s_0\} \\ \Omega_\ge &= \Omega - \Omega_< \end{aligned}$$

From condition (13), we conclude that:

$$\int_{\Omega_<} F(x,v) \ge -C_1 \int_{\Omega_<} (|v|^q + 1) \ge -C_1 \int_\Omega (s_0^q + 1) = -C_1(s_0^q + 1)|\Omega| = K_1$$

and by (14), we have that:

$$\int_{\Omega_\ge} F(x,v) \ge \int_{\Omega_\ge} \gamma(x)|v|^\theta$$

Then by (12):

$$J(v) = \int_\Omega A(x, \nabla v) - \left(\int_{\Omega_<} F(x,v) + \int_{\Omega_\ge} F(x,v) \right)$$

$$J(v) \le c_2\, \|v\|^p_{W_0^{1,p}(\Omega)} + c_2 \|\nabla v\|_1 - \int_{\Omega_\ge} \gamma(x)|v|^\theta - K_1 =$$

$$\le c_2\, \|v\|^p_{W_0^{1,p}(\Omega)} + c_2 |\Omega|^{1/p'}\, \|v\|_{W_0^{1,p}(\Omega)} - \int_\Omega \gamma(x)|v|^\theta + \int_{\Omega_<} \gamma(x)|v|^\theta - K_1$$

$$\leq c_2 \,\|v\|^p_{W_0^{1,p}(\Omega)} + c_3\|v\|_{W_0^{1,p}(\Omega)} - \int_\Omega \gamma(x)|v|^\theta + K$$

with $K = \|\gamma\|_{L^\infty}\, s_0^\theta|\Omega| - K_1$ and $c_3 = c_2|\Omega|^{1/p'}$, so (15) is proved. The functional $\|\cdot\|_\gamma : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\|v\|_\gamma = \left(\int_\Omega \gamma(x)|v|^\theta\right)^{1/\theta}$$

is a norm in $W_0^{1,p}(\Omega)$. In the finite dimensional subspace X_1 the norms $\|\cdot\|_{W_0^{1,p}(\Omega)}$ and $\|\cdot\|_\gamma$ are equivalent, so there exists a constant $\tilde{K} = \tilde{K}(X_1)$ such that:

$$\|v\|_{W_0^{1,p}(\Omega)} \leq \tilde{K}\left(\int_\Omega \gamma(x)|v|^\theta\right)^{1/\theta}$$

As a consequence, in X_1 we have that:

$$J(v) \leq c_2\tilde{K}^p\,\|v\|^p_\gamma + c_3\tilde{K}\,\|v\|_\gamma - \|v\|^\theta_\gamma + K$$

Hence

$$c_2\tilde{K}^p\,\|v\|^p_\gamma + c_3\tilde{K}\|v\|_\gamma - \|v\|^\theta_\gamma + K \geq 0\,\forall v \in S$$

and since $\theta > p$, we conclude that S is bounded.

Proof of Theorem 4.1: Since the functions a and f are odd in the second argument, the functional J is even. J verifies also that $J(0) = 0$. By lemma 3.1, J satisfies the (PS) condition. From lemma 3.2, there exist constants $\alpha,\ \rho > 0$ such that $J(u) \geq \alpha$ if $\|u\|_{W_0^{1,p}(\Omega)} = \rho$. By the previous lemma $\{v \in X_1 : J(u) \geq 0\}$ is bounded whenever X_1 is a finite dimensional subspace of $W_0^{1,p}(\Omega)$. Then theorem 4.2 can be applied to the functional J. This ends the proof.

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