Chapter 8
Using the Definite Integral

Volumes of Revolution

When a region in the plane is revolved around a line, it generates a *solid of revolution*.

Consider the region bounded by the curve $y = f(x)$, the vertical lines $x = a$ and $x = b$, and the horizontal line $y = c$. If the region is rotated about the horizontal line $y = c$, the volume of revolution is

$$\int_a^b \pi (f(x) - c)^2 \, dx$$

Explanation of the volume formula

The total volume of a solid of revolution is the sum of very thin circular slices cut across the axis of rotation. The radius of the thin circular slice cut at x is $|f(x)-c|$. If the thickness of the slice is Δx, then the volume of the slice is $\pi(f(x)-c)^2\Delta x$. Now, if we let $\Delta x \to 0$ (i.e., cutting thinner and thinner slices) and add up all the volumes of the slices, the sum approaches the integral

$$\int_a^b \pi(f(x)-c)^2\,dx$$

There are many variations of this theme. Study Examples 1- 4 on P. 375-377.

Arc Length of a curve

For $a < b$, the arc length of the curve of $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

If the curve is given parametrically for $a \le t \le b$ by differentiable functions $x(t)$ and $y(t)$, then

$$\text{Arc Length} = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Polar Coordinates (Section 8.3)

Any point on the two dimensional plane can be represented in polar coordinates (r, θ). The number r represents the length of the line segment joining the point to the origin; and θ represents the angle between this line segment and the positive x-axis measured anticlockwise.

Relation between Cartesian coordinates (x, y) and Polar coordinates (r, θ):

$$x = r\cos\theta, y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}, \tan\theta = y / x \text{ for } x \neq 0$$

Note: Generally it is not always true that $\theta = \arctan(y / x)$. The exact value of θ is determined according to the signs of x and y.

A two dimensional curve can be described by an equation in polar coordinates $r = f(\theta)$ just like it can be described by an equation in Cartesian coordinates $y = f(x)$. See P.384-385 for examples of curves which can be conveniently described by equations in polar coordinates.

Some two dimensional regions can be described conveniently by one or more inequalities in polar coordinates (e.g., the unit disc can be described as $r \leq 1$, the first quadrant as $0 \leq \theta \leq \pi / 2$). See Ex. 17-19 on P.389 for more examples.

Area in Polar Coordinates

For a curve $r = f(\theta)$, with $\alpha \le \theta \le \beta$, which does not cross itself,

$$\text{Area of region enclosed} = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 \, d\theta$$

Arc Length in Polar Coordinates

The arc length of a curve can be expressed as

$$\text{Arc Length} = \int_{\alpha}^{\beta} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} \, d\theta$$

Density and Center of Mass (Section 8.4)

To find total quantity from density, divide the region into small pieces such that the density is approximately constant on each piece, and add the contributions of the pieces.

The center of mass of a system of *n* points masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$ along the *x*-axis is given by

$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}$$

The center of mass of an object lying long the *x*-axis between $x = a$ and $x = b$ is

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx},$$

where $\delta(x)$ is the density (mass per unit length) of the object.

Applications of the Definite Integral in Economics (Section 8.6)

Future and Present values

The *future value*, $\$B$, of a payment $\$P$, is the amount to which the $\$P$ would have grown if deposited in an interest bearing bank account.

The *present value*, $\$P$, of a future payment, $\$B$, is the amount which would have to be deposited in a bank account today to produce exactly $\$B$ in the account at the relevant time in the future.

Annual Compounding
With an interest rate of *r*, compounded annually, and a time period of *t* years, a deposit of $*P* grows to a future balance $*B*, where

$$B = P(1+r)^t, \text{ or equivalently, } P = \frac{B}{(1+r)^t}$$

Continuous Compounding
If instead of annual compounding, we have continuous compounding, we get

$$B = Pe^{rt}, \text{ or equivalently, } P = Be^{-rt}$$

Income Stream

Consider a continuous income stream at the rate of $P(t)$ dollars/year being deposited in the period from now until M years in the future.

$$\text{Present Value } = \int_0^M P(t)e^{-rt}dt \text{ dollars}$$

$$\text{Future Value} = \int_0^M P(t)e^{r(M-t)}dt \text{ dollars}$$