Chapter 9
Sequences and Series

Sequences (Section 9.1)
A sequence is an infinite list of numbers

$$s_1, s_2, \ldots, s_n, \ldots$$

The terms of a sequence can be represented numerically, graphically, algebraically, or verbally.

Some sequences can also be defined recursively, by giving an equation relating the n^{th} term to the previous terms and as many of the first terms as are needed to get started.

Bounded and Convergent sequences

A sequence $s_1, s_2, \ldots, s_n, \ldots$ is *bounded* if there are real numbers K and M such that $K \le s_n \le M$ for all terms.

A sequence $s_1, s_2, \ldots, s_n, \ldots$ is *convergent* to the limit L if s_n is as close to L as we please whenever n is sufficiently large. If no finite limit L exists, we say that the sequence is *divergent*.

Theorem

(a) A convergent sequence is bounded.

(b) A bounded, monotone sequence is convergent.

Series (Section 9.2 and 9.3)

A series is a sum of numbers

$$a_1 + a_2 + \cdots + a_n + \cdots$$

where $a_1, a_2, \cdots, a_n \cdots$ are called the terms of the series. If the series contains a finite (resp. an infinite) number of terms, it is called a finite (resp. an infinite) series.

Convergence of Series

The list of numbers

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$\vdots$$

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$\vdots$$

is called the *sequence of partial sums* for the series.

We say that the infinite series converges (resp. diverges) if its sequence of partial sum converges (resp. diverges).

Geometric Series

A series of the form

$$a + ax + ax^2 + \cdots + ax^n + \cdots$$

is called a geometric series.

The sum of a finite geometric series is given by

$$S_n = \frac{a(1-x^n)}{(1-x)}, \text{ provided } x \neq 1$$

(Question: What is S_n when $x = 1$?)

An infinite geometric series converges to the sum

$$S = \frac{a}{1-x}, \text{ provided } |x| < 1.$$

If $|x| \geq 1$, the series diverges.

Class exercise for 10/26/06

Look at the infinite series in Problems # 2, 4, 6, 8, 10 on P. 448. For each series

(a) State whether it is geometric or not, and explain why.

(b) If it is geometric,
(i) write down the common ratio between consecutive terms.
(ii) determine whether the series is convergent or divergent. Explain why. If convergent, determine the sum.

Theorem 9.2: Convergence Properties of Series

Property 1

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and if k is a constant, then

- $\sum_{n=1}^{\infty} a_n \pm b_n$ converges to $\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$
- $\sum_{n=1}^{\infty} ka_n$ converges to $k\sum_{n=1}^{\infty} a_n$

Question:

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, can you tell whether $\sum_{n=1}^{\infty} a_n \pm b_n$ would diverge? How about $\sum_{n=1}^{\infty} ka_n$?

Property 2

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Property 3

Changing a finite number of terms in a series does not change whether or not it converges, although it may change the value of its sum if it does converge.

Theorem 9.3: The integral Test

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive for $x \geq c$.

- If $\int_c^{\infty} f(x)dx$ converges, then $\sum a_n$ converges.
- If $\int_c^{\infty} f(x)dx$ diverges, then $\sum a_n$ diverges.

Corollary to Theorem 9.3

The p-series $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$and diverges if $p \leq 1$.

More tests for convergence (Section 9.4)

Theorem 9.4: Comparison Test

Suppose $0 \le a_n \le b_n$ for all n.

- If $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Theorem 9.5: Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{where } c > 0,$$

then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Theorem 9.6: Convergence of Absolute Values Implies Convergence

If $\sum |a_n|$ converges, then so does $\sum a_n$.

We say that the series $\sum a_n$ is:

- *absolutely convergent if* $\sum a_n$ and $\sum |a_n|$ both converge.
- *conditionally convergent if* $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem 9.7: The Ratio Test

For a series $\sum a_n$, suppose the sequence of ratios $|a_{n+1}|/|a_n|$ has a limit:

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=L$$

- If $L<1$, then $\sum a_n$ converges.
- If $L>1$, or if L is infinite, then $\sum a_n$ diverges.
- If $L=1$, the test does not tell us anything about the convergence of $\sum a_n$.

Theorem 9.8 and 9.9: Alternating Series Test

A series of the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n + \cdots$$

converges to a finite limit S if

$$0 < a_{n+1} < a_n \text{ for all } n \text{ and } \lim_{n\to\infty} a_n = 0.$$

Furthermore, $\left|S - S_n\right| < a_{n+1}$.

Power Series and Intervals of Convergence (Section 9.5)

A power series about $x = a$ is a sum of constants times powers of $x - a$:

$$S(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n + \cdots$$
$$= \sum_{n=0}^{\infty} C_n (x-a)^n$$

For a fixed value of x, $S(x)$ may converge or diverge depending on the value of $x - a$.

Each power series falls into one of the following three cases, characterized by its *radius of convergence*.

- The series converges only for $x = a$: the radius of convergence is 0.
- The series converges for all values of x; the radius of convergence is ∞.
- There is a positive finite number R such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$. The radius of convergence is R and the interval of convergence is the interval of real numbers between $a-R$ and $a+R$, including any endpoint where the series converges.

Computing Radius of Convergence

Let $a_n = C_n(x-a)^n$

- If $\lim\limits_{n\to\infty} |a_{n+1}|/|a_n|$ is infinite, then $R = 0$.
- If $\lim\limits_{n\to\infty} |a_{n+1}|/|a_n|$ = 0, then $R = \infty$.
- If $\lim\limits_{n\to\infty} |a_{n+1}|/|a_n| = K|x-a|$, where K is finite and nonzero, then $R = 1/K$.

Announcements on 11/14/06

1. Test on Thursday 11/16 will be in LART 222. Please pay attention to UTEP policy on academic dishonesty.

2. Extra Office hours on Thursday 11/16 8:45-10:00 a.m.

3. **StatFest 2006 on Saturday 11/18, 8:30 a.m. – 4:30 p.m., Union Building, Suites 312E & 313E. –** up to 3 points of extra credit: 1 point for notes taken at each talk (excluding the opening, welcome, and closing remarks). Notes for a talk must contain the Speaker's name, Institution Affiliation, Title of talk, key points of the talk. Turn in your notes on Tuesday (11/21) for extra credit points.

Class Exercise for 11/14/06

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