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A GENERAL PROCEDURE FOR DERIVING DISTRIBUTIONS

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ABSTRACT

A general procedure for deriving the exact and asymptotic distributions of a certain class of test statistics in multivariate analysis is proposed. The method is based on an asymptotic expansion of gamma ratios in terms of generalized Bernoulli polynomials. The exact and asymptotic results are obtained and the method is illustrated in the problem of testing linear hypotheses in the multinormal case. In this problem the method yields Box's (1949) expansion as a special case.

I. INTRODUCTION

There are a large number of methods for getting exact and approximate null and non-null distributions of test statistics for testing various hypotheses connected with one or more multinormal populations. The following are pertinent among those concerning exact null distributions: Direct integration methods in particular cases (Wilks 1935), successive convolutions (Schatzoff 1966), the generalized partial fractions technique (Mathai and Rathie 1971), and the method of inverse Mellin transform (Mathai 1971,

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Consul 1966, Nair 1938). Among well known approximations are those of Rao (1948), and Box (1949).

In this paper we present a method for deriving exact and approximate null distributions of multivariate test criteria whose h-th null moments are given by:

$$E(u^h) = C \prod_{j=1}^{p} \frac{\Gamma[\frac{n}{2}(1+h) + a_j]}{\Gamma[\frac{n}{2}(1+h) + b_j]}, \quad C = \prod_{j=1}^{p} \frac{\Gamma[\frac{n}{2} + b_j]}{\Gamma[\frac{n}{2} + a_j]} \qquad (1.1)$$

where n is an increasing function of the sample size and a_j and b_j are bounded constants. It is assumed that there exists at least one way of pairing a_j and b_j such that $b_j - a_j > 0$ for all j. A number of criteria have moments of the above form. (cf. Sec. 4). In Sections 2 and 3 the exact density and distribution function of u are derived by asymptotically expanding the gamma ratios in (1.1) and applying the inverse Mellin transform. Section 4 contains an application to the problem of testing linear hypotheses where the asymptotic results reduce to Box's (1949) expansion.

2. ASYMPTOTIC EXPANSIONS

The procedure is based on the following well-known asymptotic expansion of the gamma ratio, see for example Luke Vol I, 1969, p. 33. For $|\arg(z+a_j)| \leq \pi-\varepsilon$, $\varepsilon > 0$ and $(a)_n = a(a+1)\ldots(a+n-1)$,

$$\frac{\Gamma(z + a_j)}{\Gamma(z + b_j)} = z^{a_j - b_j} \sum_{r=0}^{\infty} (-1)^r \frac{B_r^{(a_j-b_j+1)}(a_j)(b_j - a_j)_r}{r! z^r} \qquad (2.1)$$

where $z\to\infty$, a_j and b_j are bounded constants $b_j - a_j > 0$ and $B_r^{(a)}(x)$ is the generalized Bernoulli polynomial of order r. The Bernoulli polynomials satisfy the relation, see Luke Vol I, 1969, p. 18

$$\frac{t^a e^{xt}}{(e^t-1)^a} = \sum_{r=0}^{\infty} \frac{t^r}{r!} B_r^{(a)}(x), \quad |t| < 2\pi. \qquad (2.2)$$

The asymptotic results up to the order $O(n^{-3})$ can be illustrated by using the first three Bernoulli polynomials. These are

$$B_0^{(a)}(x) = 1,\ B_1^{(a)}(x) = x - \frac{a}{2},\ B_2^{(a)}(x) = x^2 - ax + \frac{a(3a-1)}{12}. \tag{2.3}$$

On multiplying the p gamma ratios in (2.1) and substituting $z = \frac{n}{2}(1+h)$, we obtain, for $|\arg(z+a_j)| \le \pi-\varepsilon$, $\varepsilon > 0$, $j = 1,2,\ldots,p$,

$$\prod_{j=1}^{p} \frac{\Gamma[\frac{n}{2}(1+h) + a_j]}{\Gamma[\frac{n}{2}(1+h) + b_j]} = [\tfrac{n}{2}(1+h)]^{\sum_j (a_j-b_j)} \sum_{r=0}^{\infty} \sum_{r_1+\ldots,r_p = r} (-1)^r$$

$$\times \frac{\prod_{j=1}^{p} \{B_{r_j}^{(a_j-b_j+1)}(a_j)(b_j-a_j)_{r_j}\}}{r_1!\ldots r_p!\ [\frac{n}{2}(1+h)]^r} \tag{2.4}$$

where $\sum_j (a_j-b_j) < 0$ and $\frac{n}{2}(1+h) \to \infty$. This is obtained by multiplying the p-asymptotic series of the r·h·s of (2.1) and using the result

$$\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \ldots \sum_{r_p=0}^{\infty} (\quad) \equiv \sum_{r=0}^{\infty} \sum_{r_1+\ldots+r_p=r} (\quad).$$

Similarly we obtain the expansion of the product of the gamma ratios involved in C of (1.1). We obtain

$$C = (\tfrac{n}{2})^{\sum_j (b_j-a_j)} . C' \tag{2.5}$$

where

$$C' = \sum_{r=0}^{\infty} (-1)^r (\tfrac{n}{2})^{-r} \sum_{r_1+\ldots+r_p=r} \prod_{j=1}^{p} \{\frac{B_{r_j}^{(b_j-a_j+1)}(b_j)(a_j-b_j)_{r_j}}{r_j!}\} \tag{2.6}$$

On using (2.4) - (2.6), the moment-expression (1.1) becomes

$$Eu^h = C' \sum_{r=0}^{\infty} (-1)^r (\tfrac{n}{2})^{-r} (1+h)^{-(r+\rho)} . \quad \gamma_r \tag{2.7}$$

where

$$\rho = \sum_{j=1}^{p} (b_j - a_j) > 0, \tag{2.8}$$

$$\gamma_r = \sum_{r_1+\ldots+r_p=r} \prod_{j=1}^{p} \frac{B_{r_j}^{(a_j-b_j+1)}(a_j)(b_j-a_j)_{r_j}}{r_j!} \tag{2.9}$$

2.1. THE DENSITY FUNCTION

The density is obtained from (2.7) by taking the inverse Mellin transform and by using the following lemma.

<u>Lemma 1</u>

$$\int_0^1 \frac{x^h(-\log x)^{\beta-1}}{\Gamma(\beta)}\, dx = (1+h)^{-\beta}, \quad 0<x<1, \quad \beta>0.$$

On using the Lemma, the inverse Mellin Transform of $(1+h)^{-(r+\rho)}$ is

$$\frac{-\log (u)^{r+\rho-1}}{\Gamma(r+\rho)} = \frac{(-\log u)^{\rho-1}}{\Gamma(\rho)} \cdot \frac{(-\log u)^r}{(\rho)_r}.$$

Hence, the density is obtained as follows.

$$f(u) = C' \frac{(-\log u)^{\rho-1}}{\Gamma(\rho)} \sum_{r=0}^{\infty} (-1)^r \left(\frac{n}{2}\right)^{-r} \gamma_r \frac{(-\log u)^r}{(\rho)_r}, \quad 0<u<1 \tag{2.10}$$

where

$$C' = \sum_{r=0}^{\infty} (-1)^r \left(\frac{n}{2}\right)^{-r} \delta_r$$

$$\delta_r = \sum_{r_1+\ldots+r_p=r} \prod_{j=1}^{p} \frac{B_{r_j}^{(b_j-a_j+1)}(b_j)(a_j-b_j)_{r_j}}{r_j!}. \tag{2.11}$$

Thus we have shown the following:

Theorem 1. If the h-th moment of a criterion u, $0<u<1$ is given by (1.1), where a_j, b_j, $j=1,\ldots,p$ are bounded and free of h and $\sum_j (b_j - a_j) > 0$, then the density of u is given by (2.10).

2.2. THE DISTRIBUTION FUNCTION

The distribution function is available from (2.10) by term-by-term integration using the following lemma

Lemma 2

$$\int_0^x (-\log \lambda)^{\beta-1} d\lambda = \int_{-\log x}^{\infty} y^{\beta-1} e^{-y} dy, \ 0<x<1, \ \beta>0.$$

Thus $F(x) = P_r\{u \le x\}$ is available in terms of cumulative probabilities of chi-square random variables.

$$F(x) = C' \sum_{r=0}^{\infty} (-1)^r \left(\frac{n}{2}\right)^{-r} \left(\frac{1}{\Gamma(r+\rho)} \int_0^x (-\log u)^{r+\rho-1} du\right) \gamma_r$$

$$= C' \sum_{r=0}^{\infty} (-1)^r \left(\frac{n}{2}\right)^{-r} \gamma_r I_r(x) \qquad (2.12)$$

where

$$I_r(x) = 1 - \frac{1}{\Gamma(r+\rho)} \int_0^{-\log x} y^{r+\rho-1} e^{-y} dy$$

$$= 1 - \Pr(\chi^2_{2(r+\rho)} \le -2 \log x), \ r=0, 1, \ldots \qquad (2.13)$$

It should be noted that one advantage of the representations in (2.10) and (2.12) is that any desired degree of accuracy can be obtained since γ_r and δ_r in (2.9) and (2.11) can be easily programmed for all values of r. The required generalized Bernoulli polynomials are easily derived from the recursive formula, see Luke Vol 1, (1969) p. 22

$$B_{k+1}^{(a)}(x) = B_1^{(a)}(x) B_k^{(a)}(x) - a \sum_{r=0}^{[(k-1)/2]} \frac{\binom{k}{2r+1} B_{2r+2}}{2r+2} B_{k-2r-1}^{(a)}(x), \qquad (2.14)$$

where B_{2r+2} are the Bernoulli numbers, see the above reference page 20 for B_{2r+2}, $r=0,1,\ldots,8$.

Explicit expressions for γ_i, δ_i, $i=1,2$ are given below where we have put $c_j=a_j-b_j+1$ and $d_j=b_j-a_j+1$.

$$\gamma_1 = \sum_{j=1}^{p} B_1^{(c_j)}(a_j)(b_j-a_j)$$

$$\delta_1 = \sum_{j=1}^{p} B_1^{(d_j)}(b_j)(a_j-b_j) = -\gamma_1 \tag{2.15}$$

$$\gamma_2 = \frac{1}{2}\sum_{j=1}^{p} B_2^{(c_j)}(a_j)(b_j-a_j)_2$$

$$+ \sum_{j<k} B_1^{(c_j)}(a_j)B_1^{(c_k)}(a_k)(b_j-a_j)(b_k-a_k) \tag{2.16}$$

$$\delta_2 = \frac{1}{2}\sum_{j=1}^{p} B_2^{(d_j)}(b_j)(a_j-b_j)_2$$

$$+ \sum_{j<k} B_1^{(d_j)}(b_j)B_1^{(d_k)}(b_k)(a_j-b_j)(a_k-b_k). \tag{2.17}$$

Hence, an approximation of $F(x)$ up to order $O(n^{-3})$ is easily obtained as

$$F(x) = 1 - \Pr(X^2_{2\rho} \le -2\log x) - \frac{2\gamma_1}{n}[\Pr(X^2_{2\rho} \le -2\log x)$$

$$- \Pr(X^2_{2(\rho+1)} \le -2\log x)] + \frac{4}{n^2}[-\delta_2\cdot\Pr(X^2_{2\rho} \le -2\log x)$$

$$-\gamma_2\cdot\Pr(X^2_{2(\rho+2)} \le -2\log x) - \gamma_1\delta_1\cdot\Pr(X^2_{2(\rho+1)} \le -2\log x)$$

$$+ \gamma_2 + \delta_2 + \gamma_1\delta_1] + O(n^{-3}). \tag{2.18}$$

3. MODIFICATIONS

The present method is also applicable to different variations of (1.1). Suppose there was a factor a^h on the right side of (1.1) where a is free of h. Then the only change will be to replace the variable u by u/a on the right side of $f(u)$, because in the inverse Mellin transform we have the factor $a^h u^{-h} = (u/a)^{-h}$ instead of u^{-h}.

In (1.1) if there are $n_1, n_2, \ldots, n_p$ instead of n, then the only change will be to replace $(n/2)^{-\Sigma(b_j - a_j)}$ by $\prod_{j=1}^{p} (n_j/2)^{-(b_j - a_j)}$ and $(n/2)^{-r}$ by $\prod_{j=1}^{p} (n_j/2)^{-r_j}$ in (2.4) and make the corresponding changes in all other equations from (2.4) onwards.

But the more interesting modification one may wish to introduce is to incorporate some arbitrary constants such that the series form for F(x) can be made to converge faster or to make the computations simpler. This can be done in two ways.

1) Suppose that we wish to introduce an arbitrary parameter so that $\gamma_1 = -\delta_1$ is made zero, so that several terms can be deleted from (2.18). We proceed in the following way. In terms of the inverse Mellin transform the explicit expression for f(u) is the following

$$f(u) = C\ (2\pi i)^{-1} \int_L \prod_{j=1}^{p} \frac{\Gamma[\frac{n}{2}(1+h)+a_j]}{\Gamma[\frac{n}{2}(1+h)+b_j]} u^{-h}\, dh \tag{3.1}$$

where L is a suitable contour and $i=(-1)^{1/2}$. Put $h = h' - \lambda/n$. Then,

$$f(u) = C\ u^{\lambda/n} (2\pi i)^{-1} \int_{L'} \prod_{j=1}^{p} \frac{\Gamma[\frac{n}{2}(1+h')+a_j-\lambda/2]}{\Gamma[\frac{n}{2}(1+h')+b_j-\lambda/2]} u^{-h'} dh'. \tag{3.2}$$

The only changes are that a_j and b_j are replaced by $a_j-\lambda/2$ and $b_j-\lambda/2$ respectively, and the factor $u^{\lambda/n}$ comes out of the integral. Thus, the density in (2.10) has an extra factor $u^{\lambda/n}$ and hence the integral in (2.12) is to be modified to the following.

<u>Lemma 3</u>

$$\int_0^x u^{\lambda/n} (-\log u)^{\rho+r-1} du$$

$$= (1+\lambda/n)^{-(\rho+r)} \int_{-(1+\lambda/n)\log x}^{\infty} u^{\rho+r-1} e^{-u} du$$

Under this modification, the new γ_1 (see (2.15)) becomes

$$\gamma_1 = \frac{1}{2}\sum_{j=1}^{p}(b_j-a_j)(b_j+a_j-1-\lambda). \tag{3.3}$$

Now choose λ such that $\gamma_1=0$ which will knock out a number of terms from (2.18).

2) Suppose that the expansion in (2.12) is desired in powers of m^{-1} (say), instead of n^{-1}, where $n=m-\lambda$ and λ is chosen so that $\gamma_1=-\delta_1=0$. This is possible with some criteria. Such a modification is particularly interesting in the application discussed in the next section, since it reduces (2.12) to Box's (1949) expansion with coefficients easier to compute.

4. APPLICATIONS

A structure of the type (1.1) appears in a number of likelihood ratio test statistics in multivariate analysis. The h-th null moments of one to one functions of the following test statistics have either the structure (1.1) or they can be easily transformed to the form (1.1): testing linear hypotheses, equality of covariance matrices, equality of populations, sphericity, independence, compound symmetry, Wilks' L_{vc}, Wilks' L_{mvc}, and certain tests on eigenvalues connected with multinormal populations. Certain volume contents of random geometric configurations also have h-th moments which can be transformed to the form (1.1), see for example Mathai (1982). For the sake of illustration we will consider the h-th null moment coming from testing linear hypotheses in the multinormal case.

Let λ be the criterion for testing the hypotheses that a submatrix $B_1(p \times q)$ of the matrix of the regression coefficients $B(p \times r)$ is equal to a specified matrix. The h-th null moment of $U = \lambda^{2/N}$ where N is the sample size is (Anderson 1958, p. 192),

$$E(U^h) = C\prod_{j=1}^{p}\frac{\Gamma[\frac{n}{2}+h+\frac{1-j}{2}]}{\Gamma[\frac{n}{2}+h+\frac{q+1-j}{2}]} \tag{4.1}$$

where $n = N-r$, p is the dimensionality of the population, and

$$C = \prod_{j=1}^{p}\frac{\Gamma[\frac{n}{2}+\frac{q+1-j}{2}]}{\Gamma[\frac{n}{2}+\frac{1-j}{2}]} \tag{4.2}$$

The exact null distribution of $U = U_{p,q,n}$ is obtained by several authors. Wilks (1935) obtained the exact distributions for the cases p=1,2,3, q=3; p=4, q=4 by direct methods. Consul (1966) also derived exact distributions for the particular cases p=1,2,3,4 and q=3,4,5,6,7,8 by using the technique of Mellin transforms. Exact distributions for general p and q have been obtained by Schatzoff (1964, 1966), Pillai and Gupta (1969), Mathai and Rathie (1971), and Lee 1972b. Approximations and asymptotic expansions are given by Bartlett (1938), Rao (1948), and Box (1949).

We now apply the present method with the modification 2) of Section 3, i.e. put $n=m-\lambda$ in (4.1) and (4.2) and then mh/2 for h to obtain the moments of $u_*=U^{m/2}$,

$$E(u_*^h) = \prod_{j=1}^{p} \frac{\Gamma[\frac{m}{2} + \frac{q-\lambda+1-j}{2}]}{\Gamma[\frac{m}{2} + \frac{1-j-\lambda}{2}]} \prod_{j=1}^{p} \frac{\Gamma[\frac{m}{2}(1+h) + \frac{1-j-\lambda}{2}]}{\Gamma[\frac{m}{2}(1+h) + \frac{q-\lambda+1-j}{2}]} .$$

The above is of the form (1.1) with n replaced by m and

$$a_j = \frac{1-j-\lambda}{2}, \quad b_j = \frac{q-\lambda+1-j}{2} \tag{4.3}$$

On requiring $\gamma_1=0=-\delta_1$, from (2.15) we have

$$\sum_{j=1}^{p} (b_j-a_j)(b_j+a_j-1-\lambda) = 0 \tag{4.4}$$

(see also (3.3)). On substituting from (4.3) we obtain

$$\lambda = (q-p-1)/2, \tag{4.5}$$

and hence $m=n-(p-q+1)/2$ and

$$a_j = (p-q+3-2j)/4, \quad b_j = (p+q+3-2j)/4. \tag{4.6}$$

For the above values of a_j and b_j, all conditions of Theorem 1 are satisfied and the exact density of u_*, $0<u_*<1$, is given by (2.10) with n replaced by m and $\rho = \sum_j(b_j-a_j) = pq/2$. The values of γ_r, δ_r and C' are obtained from (2.9), (2.11) and (2.6).

The exact distribution function of u_*, $P_r(u_*\le x)$ is given by (2.12) where again n is replaced by m and ρ, γ_r, δ_r, and C' are as in the density. Thus, we have

$$F(x) = C' \sum_{r=0}^{\infty} (-1)^r (\tfrac{m}{2})^{-r} \gamma_r I_r(x). \tag{4.7}$$

An expansion in powers of m^{-1} of the $P_r\{-m \log U \ge z\}$ is easily obtained from (4.7) as $F(\exp(-\frac{z}{2}))$ since

$$P_r\{-m \log U \ge z\} = P_r\{u_* \le \exp(-\tfrac{z}{m})\} = F(\exp(-\tfrac{z}{2}))$$

$$= C' \sum_{r=0}^{\infty} (-1)^r (\tfrac{m}{2})^{-r} \gamma_r [1 - P_r(X^2_{2r+pq} \le z)]. \tag{4.8}$$

The above is Box's (1949) series expansion (since it is an expansion of $P_r\{-m \log U \ge z\}$ in powers of m^{-1}). For an illustration consider the first two terms of (4.8),

$$P_r\{-m \log U \ge z\} = 1 - P_r\{X^2_{pq} \le z\}$$

$$+ \frac{4}{m^2}[\gamma_2 + \delta_2 - \delta_2 \cdot P_r\{X^2_{pq} \le z\} - \gamma_2 P_r\{X^2_{pq+4} \le z\}] + O(m^{-3}). \tag{4.9}$$

Using the values of a_j and b_j from (4.6), (2.16) and (2.17) give

$$\gamma_2 = -\delta_2 = pq(p^2+q^2-5)/192$$

and hence (4.9) gives the first two terms of Box's expansion, see Anderson (1958), p. 208. It should be noted however that the advantage of the representation (4.8) is that any number of terms can be easily computed.

The method easily yields an expansion of $P_r\{-m \log U \ge z\}$ in a series in powers of n^{-1}. In this case it is seen from (4.1) and (4.2) that $a_j = (1-j)/2$, $b_j = (q+1-j)/2$. On using (2.15) - (2.17), the approximation (2.18) is easily written down. This approximation is of the same accuracy as Box's (1949) of order $O(N^{-6})$ (see Anderson (1958), p. 208) for $n \ge 30$, as was shown by numerical comparisons not reported here; for smaller values of n however, (2.18) was less accurate than Box's due to the difference in the order of approximation.

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