ASYMPTOTIC EXPANSIONS OF THE NON–NULL DISTRIBUTION
OF THE LIKELIHOOD RATIO CRITERION
FOR MULTISAMPLE SPHERICITY

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SYNOPTIC ABSTRACT

Let \( \lambda^* \) be the modified likelihood ratio criterion for testing the
multisample sphericity hypothesis \( H: \Sigma_j = \sigma^2 I, j = 1, \ldots, k \), where \( \Sigma_j \) is
the positive definite covariance matrix of the \( j^{th} \) normal population.
Asymptotic expansions for the distribution of \( \lambda^* \) are derived, under two
classes of local alternatives. The results are an extension of those in Khatri

Key Words and Phrases: Multisample sphericity; likelihood ratio test;
 asymptotic expansions; local alternatives.
1. INTRODUCTION.

Consider the repeated measurements model (1.1) with a fixed number \( p \) of responses (collected on the same experimental unit at successive times, or under a variety of experimental conditions)

\[
X_{ij} = \mu + \beta_j + e_{ij}; i = 1, \ldots, k; j = 1, \ldots, p
\]  

(1.1)

where \( \mu \) is the general-level parameter common to all observations, \( \beta_j \) is the measure of the effect specific to the \( j^{th} \) condition, and \( e_{ij} \) is a random disturbance reflecting both the experimental error and the interaction between the \( i^{th} \) unit and the \( j^{th} \) response.

Consider the hypothesis of equal response effect

\[
H_0: \beta_1 = \beta_2 = \cdots = \beta_p
\]  

(1.2)

or, in matrix form, \( C\beta = 0 \) where \( C \) is an orthogonal contrast matrix, for example

\[
C = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1
\end{bmatrix}
\]  

(1.3)

Let \( e_i \) be the vector of the disturbances corresponding to the \( i^{th} \) sampling unit, i.e., \( e_i = (e_{i1}, \cdots, e_{ip})' \). If the \( e_i \)'s have the same multivariate normal distribution \( N_p(0, \Sigma) \), then \( H_0 \) is tested by the well-known Hotelling's \( T^2 \) statistic computed from the mean vector and sample covariance matrix of the differences.
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\[ Y_{ij} = X_{ij} - X_{i,j+1}, \quad i = 1, \ldots, k; \quad j = 1, \ldots, p - 1. \]

Denoting by \( S_Y \) the \((p-1) \times (p-1)\) covariance matrix of the \( Y_{ij} \) differences, the test statistic is (see Morrison (1976))

\[
F = \frac{(k - p + 1) \, T^2}{(k - 1)(p - 1)} \quad (1.4)
\]

where

\[
T^2 = \bar{Y} \cdot S_Y^{-1} \bar{Y} \quad (1.5)
\]

\[
\bar{Y} = (\bar{Y}_1, \ldots, \bar{Y}_{p-1})' = (\bar{X}_1 - \bar{X}_2, \ldots, \bar{X}_{p-1} - \bar{X}_p)'.
\]

When \( H_0 \) is true, \( F \) follows an \( F \)-distribution with \( p - 1 \) and \( k - p + 1 \) degrees of freedom.

However, if the \( \epsilon_i \)'s have different covariance matrices, i.e., \( \epsilon_i \sim N_p(0, \Sigma_i), \quad i = 1, \cdots, k \), then the above \( F \)-ratio may not be valid. An assumption essential to its validity is (see Huynh and Feldt (1970))

\[
H_0' : \Sigma_1 C' = \Sigma_2 C' = \cdots = \Sigma_k C' = \sigma^2 I. \quad (1.6)
\]

Huynh and Feldt (1970) suggest testing \( H_0' \) in two steps: first test the hypothesis \( H_a' : \Sigma_j C' \) equal for all \( j \); and second, test the hypothesis \( H_b' : \Sigma C' = \sigma^2 I \) where \( \Sigma \) is the pooled dispersion matrix. If \( H_a' \) and \( H_b' \) are not rejected, then \( H_0' \) is assumed.

The above considerations provide the motivation for a single test of multisample sphericity based on k independent samples from multivariate normal populations. Specifically, let \( X_{ij} \{i = 1, \cdots, N_j; \quad j = 1, \cdots, k\} \) be \( k \) independent samples from normal populations \( N_p(\mu_j, \Sigma_j) \) where \( \mu_j \) is a \( p \times 1 \) real vector and \( \Sigma_j \) is a positive definite covariance matrix. The hypothesis of multisample sphericity is
\( H: \Sigma_j = \sigma^2 I, j = 1, \ldots, k \) \hspace{1cm} (1.7)

where \( \sigma^2 \) is an unknown scalar. (This is an obvious generalization of the one-sample sphericity hypothesis of Mauchly (1940).)

The hypothesis \( H \) is considered in Mendoza (1959) where the null moments of the associated likelihood ratio are given, with a one-term approximation to its null distribution. The approximation consists of a limiting chi-square term, adjusted for faster convergence by the method of Box (1949). Further study of hypothesis \( H \) is given by Mathai (1984), who derives various forms of the non-null moments of the modified likelihood ratio test for \( H \), and points out that the exact non-null distribution is complicated and may be obtained in a series of \( H \)-functions.

In the one-sample case \( (k = 1) \), the modified likelihood ratio test has been investigated extensively. The exact non-null distribution is derived in Khatri and Srivastava (1971), and various asymptotic expansions are available. An expansion under fixed alternatives is given in Sugiura (1969) in terms of a normal distribution function and its derivatives, while Khatri and Srivastava (1974) give expansions in terms of chi-squares, under alternatives that are close to the null hypothesis; also see Nagao (1970, 1973).

In this paper we consider the \( k \geq 1 \) distribution of the modified likelihood ratio test for testing \( H \), under the type of alternatives considered in Khatri and Srivastava (1974) for the \( k = 1 \) case. In particular we show that (with appropriate modifications of the notation) the results of Khatri and Srivastava can be extended to multisample sphericity. The alternatives are

\[
A_1: \Sigma_j = \sigma^2 (I + \frac{\Omega_j}{m}), j = 1, \ldots, k, \hspace{1cm} (1.8)
\]

\[
A_2: \Sigma_j = \sigma^2 (I - \frac{\Gamma_j}{m}), j = 1, \ldots, k, 0 < q < \infty, \hspace{1cm} (1.9)
\]

where \( \Omega_j \) and \( \Gamma_j \) are diagonal matrices and
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\[ m = N_1 + N_2 + \cdots + N_k - k - 2\alpha \]

where \( \alpha \) is a constant chosen so as to accelerate convergence of the asymptotic series expansions of the null and non-null distribution; its exact expression is given in Lemma 1 of Section 2.

Under the above alternatives, the distribution of the modified likelihood ratio test for \( \mathcal{H} \) is expressed in Sections 3 and 4 up to \( O(m^{-3}) \) in terms of chi-square probability distributions. (This requires a two-term approximation to the null moments, given in Lemma 1 of Section 2.) The final results are given at the ends of Sections 3 and 4. Section 5 contains a comprehensive example.

2. APPROXIMATION TO THE NULL DISTRIBUTION.

The test of \( \mathcal{H} \) of (1.7) is based on the modified likelihood ratio \( (\lambda^*)^{n/2} \) where \( n = N - k, 0 < \lambda^* < 1 \), and rejects \( \mathcal{H} \) for small values, i.e., smaller than \( \lambda_\alpha \) such that \( \Pr(\lambda^* < \lambda_\alpha) = \alpha \) when \( \mathcal{H} \) is true, where

\[
\lambda^* = \left( p_1 \left( \frac{k}{j=1} \theta_j \right)^p \frac{k}{j=1} |SS_j|^p \right)^{(\text{tr}A)^-p}
\]

(2.1)

where \( N_j - 1 = n_j = \theta_j n, \Sigma_j \theta_j = I, \) and \( A = SS_1 + \cdots + SS_k \), where \( SS_j \) is the matrix of sums of squares and cross-products for the \( j \)th sample. The \( k \)th moment of \( (\lambda^*)^{n/2} \) is derived in Mathai (1984) as

\[
E(\lambda^*)^{nh/2} = \left[ \left( \frac{n}{p} \right)^{nh/2} \prod_{j=1}^k n_j^{nh/2} \right]^h
\]

\[
\cdot \prod_{j=1}^k \left[ \left( q \theta_j \right)^{-n_j/2} \Gamma\left[\frac{n_j}{2}(1 + h)/2\right]\right]
\]

\[
\cdot \left[ \Gamma\left(\frac{nh}{2}\right) \right]^{1/2} \int_0^1 \left( 1 - y \right)^{nh/2 - 1} \left( 1 - y \right)^{nh/2 - 1}
\]
where $q$ is an arbitrary positive constant and $z_{ji} = 1 - q^{-1} \sigma_{ji}^2$ where $\sigma_{ji}$, $i = 1, \ldots, p$ are the eigenvalues of $\Sigma_j$, $j = 1, \ldots, k$. Using moment expression (2.2), Mathai obtained alternative representations by noting that the integral part is simply that of an integral representation of Lauricella's $F_D$ hypergeometric function in $p_k$ variables. By choosing the arbitrary constant $q$ so that $|z_{ji}| < 1$, the moment is expressible as a convergent multiple series.

On substituting $(2/n)h$ for $h$, then $n = m + 2\alpha$, and finally $(m/2)h$ for $h$ (in that order), we obtain

$$E(\lambda^*)^{mh/2} = \phi(h) \prod_{j=1}^k |q\Sigma_j|^{-\Theta_m/2-\Theta_\alpha} \cdot g(h)$$

where

$$g(h) = \frac{\Gamma[pm(1 + h)/2 + p\alpha]}{\Gamma[pm/2 + p\alpha]\Gamma[pm/2]} \cdot \int_0^1 y^{pm/2+p\alpha-1} (1-y)^{pm/2-1}$$

$$\cdot \prod_{j=1}^k \prod_{i=1}^p (1-z_{ji}y)^{-\Theta_m(1+h)/2-\Theta_\alpha} dy,$$

and

$$\varphi(h) = \left(\frac{p}{\prod_{j=1}^k \Theta_j}\right)^{pm/2} \cdot \prod_{j=1}^k \left\{ \frac{\Gamma[\Theta_j m(1 + h)/2 + \Theta_j \alpha]}{\Gamma[p \Theta_j m/2 + \Theta_j \alpha]} \right\}$$

$$\frac{\Gamma[pm/2 + p\alpha]}{\Gamma[pm(1 + h)/2 + p\alpha]}$$

(2.5)
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When H is true, moment (2.3) reduces to (2.5) which is the null moment. Applying the standard method of Box (1949) (also see Anderson (1984)), we obtain the following result.

Lemma 1. If

\[ \alpha = \left[ p \left( 2p^2 + 3p - 1 \right) \sum_{j=1}^{k} (1/2) \right] / \left( kp(p+1) - 2 \right) \]

then

\[ \phi(h) = \left( 1 + h \right)^{-f/2} \left\{ 1 + \gamma_2 \left( 1 + h \right)^{2/3} \right\} / m^2 + \text{O}(m^{-3}) \]

(2.6)

where

\[ f = kp(p + 1)/2 - 1 \]

\[ \gamma_2 = f \alpha^2 + S_1 \alpha + S_0 \]

\[ S_0 = [p(p - 1)(5p^2 - p - 2)/48] \sum_{j=1}^{k} \phi_j^2 \]

\[ S_1 = -[p(2p^2 + 3p - 1)/12] \sum_{j=1}^{k} \phi_j^4 + (1/3p) \cdot \alpha \]

It follows from (2.6) that

\[ \text{P}[\ln \lambda^* \leq z] = \text{P}[\chi_f^2 \leq z] + \left( \gamma_2/m^2 \right) \cdot \left[ \text{P}[\chi_f^4 + 4 \leq z] - \text{P}[\chi_f^2 \leq z] \right] + \text{O}(m^{-3}). \]

(2.7)

To assess the accuracy of the above approximation, exact percentage points for the distribution of \( \lambda^* \) are required; these are presently unavailable. For \( k = 1 \), the above reduces to the well-known approximation in the one-sample case; see Anderson (1984, p. 432).

3. THE DISTRIBUTION OF \( \lambda^* \) UNDER \( A_1 \).

Consider moment expression (2.3). Since the expansion of \( \phi(h) \) is available from Lemma 1, we need to expand the other factors. We begin by expanding \( g(h) \) under \( A_1 \).

Since \( q \) is arbitrary, it may be chosen so that \( |z_{ij}| < 1, \ i = 1, \ldots, \)
p, j = 1, \cdots, k and hence the product involving $z_{ji}$ in (2.4) may be expanded using a logarithmic expansion. The choice $q = \sigma^2$, under $A_1$ gives

$$z_{ji} = 1 - q^{-1} \sigma_j^{-1} = 1 - (1 + \omega_{ji}/m)$$

$$= \omega_{ji}/m - \omega_{ji}^2/m^2 + \omega_{ji}^3/m^3 + \cdots$$

(3.1)

where $\omega_{ji}$ are the diagonal elements of $\Omega_j$, i = 1, \cdots, p, j = 1, \cdots, k.

Then for $\min(\omega_{ji}) > -m/2$ it is $|z_{ji}| < 1$. Hence, for $0 < y < 1$ we obtain

$$R(y) = \prod_{j=1}^{k} \prod_{i=1}^{p} \left(1 - z_{ji}y\right)^{-\theta_j m(1+h)/2 - \theta_j /\alpha} = \exp\{\ln R(y)\}$$

$$e^{(1+h)t_1/y/2} \{1 + (1/m) \cdot (c_1(h)y + c_2(h)y^2)$$

$$+ (1/m^2) \sum_{j=1}^{4} d_j(h)y^j + 0(m^{-3})\}$$

(3.2)

where

$$t_i = \sum_{j=1}^{k} \theta_j \text{tr}(\Omega_j), i = 1, 2, \cdots$$

(3.3)

$$c_1(h) = \alpha t_1 - t_2(1 + h)/2$$

(3.4)

$$c_2(h) = t_2(1 + h)/4$$

(3.5)

$$d_1(h) = -\alpha t_2 + (1 + h) t_3/2$$

(3.6)

$$d_2(h) = \alpha t_2/2 + \alpha t_3/2 - (t_3 + \alpha t_2)(1 + h)/2$$

$$+ t_2^2(1 + h)^2/8$$

(3.7)
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\[ d_3(h) = \frac{t_3}{6} + \alpha_1 t_3/4)(1 + h) - t_3^2(1 + h)^2/8 \]  \hspace{1cm} (3.8)

\[ d_4(h) = t_3^2(1 + h)^2/32. \]  \hspace{1cm} (3.9)

Using (3.2), (2.4) becomes

\[
g(h) = \frac{\Gamma(p.m(1 + h)/2 + p\alpha)}{\Gamma(p.m/2 + p\alpha) \Gamma(p.mh/2)} \\
\cdot \left\{ \int_0^1 y^{p.m/2+p\alpha-1} (1 - y)^{p.mh/2-1} e^{(1+h)t_4y/2} dy \\
+ \frac{1}{m} [c_1(h) \int_0^1 y^{p.m/2+p\alpha} (1 - y)^{p.mh/2-1} e^{(1+h)t_4y/2} dy \\
+ c_2(h) \int_0^1 y^{p.m/2+p\alpha+1} (1 - y)^{p.mh/2-1} e^{(1+h)t_4y/2} dy] \\
+ \frac{1}{m^2} \sum_{j=1}^4 d_j(h) \int_0^1 y^{p.m/2+p\alpha+j-1} (1 - y)^{p.mh/2-1} \\
\cdot e^{(1+h)t_4y/2} dy + o(m^{-3}) \right\} \\
= g_0(h) + \frac{1}{m} [c_1(h) g_1(h) + c_2(h) g_2(h)] \\
+ \frac{1}{m^2} \sum_{j=1}^4 d_j(h) g_j(h) + o(m^{-3}) \] \hspace{1cm} (3.10)

where for \( \ell = 0, 1, 2, 3, 4 \), \( g_{\ell}(h) \) is given by

\[
g_{\ell}(h) = \frac{(p.m/2 + p\alpha)_{\ell}}{\Gamma(p.m(1 + h)/2 + p\alpha)_{\ell}}.
\]
\begin{equation}
\sum_{k=0}^{\infty} \frac{(pm/2 + pa + \ell)_k ((1 + h)_{1/2})^k}{[pm(1 + h)/2 + pa + \ell]_k k!}
\end{equation}

with \((a)_k = a(a + 1) \cdots (a + k - 1), (a)_0 = 1\). Equality (3.10) follows from the following integral and series representation of an \(\ _1F_1(a; c; z)\) hypergeometric function; see Erdelyi, Magnus, Oberhettinger, and Tricomi (1953), p. 255:

\[
\frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)} \int_0^1 y^{a-1} (1-y)^{c-a-1} e^{xy} dy
\]

\[
= \ _1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}, \quad \text{Re}(c) > \text{Re}(a) > 0
\]

and \(z\) is free of \(y\). In obtaining (3.10), the above series representation was used with \(a = pm/2 + pa + j, c = pmh/2 + pa + j, z = (1 + h)_{1/2} / 2\) and \(h\) such that the conditions are satisfied. An asymptotic expansion of \(g_\ell(h)\) up to the order \(0(m^{-3})\) will be obtained next.

3.1. Asymptotic Expansion of \(g_\ell(h)\).

**Lemma 2.** For \(\ell = 0,1,2,3,4\) and large \(m\) we have:

\[
\frac{(pm/2 + pa + \ell)_k}{[pm(1 + h)/2 + pa + \ell]_k} = (1 + h)^{-k} \{1 + (1/m)\}
\]

\[
\cdot A_k(\ell)[1 - (1 + h)^{-1}] + (1/m^2)[B_k(\ell) + G_k(\ell) \cdot (1 + h)^{-1}
\]

\[
+ H_k(\ell) \cdot (1 + h)^{-2}] + 0(m^{-3}),
\]

(3.12)

where \(\gamma_\ell = 2p\alpha + 2\ell\),

\[
A_k(\ell) = k(\gamma_\ell + k - 1)/p;
\]
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\[ B_K(\ell) = \frac{1}{6p^2} \{ 3k^4 + (6\gamma_k - 10)k^3 + (3\gamma_k - 12\gamma_k + 9)k^2 \}
- (3\gamma_k - 6\gamma_k + 2)k; \]

\[ G_k(\ell) = -\{ k^4 + 2(\gamma_k - 1)k^3 + (\gamma_k - 1)2k^2 \}/p^2; \]

\[ H_k(\ell) = -(B_k(\ell) + G_k(\ell)). \]

**Proof:** The proof is based on the following well-known asymptotic expansion of the ratio of two gamma functions; see for example Luke (1969), p. 33:

\[ \frac{\Gamma(z + a)}{\Gamma(z + b)} = 2^{a-b} \left[ \frac{\Gamma(a-b)(a+b-1)}{2} + (a-b)(a+b-1) \right]
\cdot \left\{ 3[(a+b-1)^2 - a + b - 1] \right\}[1 + O(z^3)] \]

where a and b are bounded constants, b = a > 0 and z → ∞. Using the above, we have:

\[ \frac{(pm/2 + p\alpha + \ell)_k}{[pm(1 + h)/2 + p\alpha + \ell]_k} = \frac{\Gamma(pm/2 + p\alpha + \ell + k)}{\Gamma(pm/2 + p\alpha + \ell)\Gamma(pm(1 + h)/2 + p\alpha + \ell + k)} \]

\[ = (pm/2)^k[1 + (2p\alpha + 2\ell + k - 1)] \cdot \frac{k(\ell + k - 1)}{6p^2m^2} \]

\[ \cdot [3(2p\alpha + 2\ell + k - 1)2 - k - 1)][1 + O(m^{-3})] \]

\[ = (pm/2)^{-k}(1 + h)^{-k}[1 - \frac{k(2p\alpha + 2\ell + k - 1)}{pm(1 + h)}] \]

\[ + \frac{k(1 + h)}{6p^2m^2(1 + h)^2} \cdot [3(2p\alpha + 2\ell + k - 1)^2 + k - 1)] \]}
\[
\cdot [1 + 0(m^{-3})],
\]

and the lemma follows after some algebra. \(\Box\)

**Lemma 3.** For \(\ell = 0, 1, 2, 3, 4\) and large \(m\), we have:

\[
\sum_{k=0}^{\infty} \frac{(pm/2 + p\alpha + \ell)_k}{[pm(1 + h)/2 + p\alpha + \ell]_k} \frac{[(1 + h)t_{1/2}]^k}{k!} = e^{t_{1/2}} \cdot \{1 + (1/m) \cdot C(\ell)[1 - (1 + h)^{-1}] + (1/m^2)[D_1(\ell) + D_2(\ell) \cdot (1 + h)^{-1} + D_3(\ell) \cdot (1 + h)^{-2}]) + o(m^{-3}),
\]

where

\[
C(\ell) = [(t_{1/2})^2 + \gamma_{\ell} t_{1/2}]/p;
\]

\[
D_1(\ell) = (1/6p^2) \{3(t_{1/2})^4 + (6\gamma_{\ell} + 8)(t_{1/2})^3 + (3\gamma_{\ell}^2 + 6\gamma_{\ell})(t_{1/2})^2\};
\]

\[
D_2(\ell) = (-1/p^2) \{(t_{1/2})^4 + (2\gamma_{\ell} + 4)(t_{1/2})^3 + (\gamma_{\ell}^2 + 2\gamma_{\ell} + 2)(t_{1/2})^2 + \gamma_{\ell}^2 t_{1/2}\}
\]

\[
D_3(\ell) = -(D_1(\ell) + D_2(\ell))
\]

and \(\gamma_{\ell} = 2p\alpha + 2\ell\).

**Proof:** On substituting (3.12) in the left-hand side of (3.13), it becomes:
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\[ \sum_{k=0}^{\infty} \frac{(pm/2 + p\alpha + \ell)_{k}}{(pm(1 + h))/2 + p\alpha + \ell}_{k!} (1 + h)_{1/2}^{k} \]

\[ = e^{t_{1}/2} \left\{ 1 + (1/m)[1 - (1 + h)^{-1}] \sum_{k=0}^{\infty} A_{k}(\ell) \frac{(t_{1}/2)^{k}}{k!} \right. \]

\[ + (1/m^{2}) \left[ \sum_{k=0}^{\infty} B_{k}(\ell) \frac{(t_{1}/2)^{k}}{k!} + (1 + h)^{-1} \sum_{k=0}^{\infty} G_{k}(\ell) \frac{(t_{1}/2)^{k}}{k!} \right] \]

\[ + (1 + h)^{-2} \sum_{k=0}^{\infty} H_{k}(\ell) \frac{(t_{1}/2)^{k}}{k!} \} \right\} + o(m^{-3}). \]

The lemma now follows by summing the above series using the formulas

\[ k^{2} = k(k - 1) + k; \]

\[ k^{3} = k(k - 1)(k - 2) + 3k(k - 1) + k; \]

\[ k^{4} = k(k - 1)(k - 2)(k - 3) + 6k(k - 1)(k - 2) + 7k(k - 1) + k; \]

\[ \sum_{k=0}^{\infty} k(t_{1}/2)^{k}/k! = (t_{1}/2)e^{t_{1}/2}; \]

\[ \sum_{k=0}^{\infty} k^{2}(t_{1}/2)^{k}/k! = [(t_{1}/2)^{2} + t_{1}/2]e^{t_{1}/2}; \]

\[ \sum_{k=0}^{\infty} k^{3}(t_{1}/2)^{k}/k! = [(t_{1}/2)^{3} + 3(t_{1}/2)^{2} + (t_{1}/2)]e^{t_{1}/2}; \]

\[ \sum_{k=0}^{\infty} k^{4}(t_{1}/2)^{k}/k! = [(t_{1}/2)^{4} + 6(t_{1}/2)^{3} + 7(t_{1}/2)^{2} + t_{1}/2]e^{t_{1}/2}. \]

With the help of Lemmas 2 and 3, we can now obtain the asymptotic expansion of \( g_{\ell}(h) \) in (3.11). Application of (3.12) with \( \ell = 0 \) and \( k = \ell \) gives:
\[ \frac{(pm/2 + p_0) \ell}{pm(1 + h)/2 + po\ell} = (1 + h)^{-\ell}(1 + (1/m)A_\ell(0)) \]

\[ \cdot [1 - (1 + h)^{-1}] + (1/m\ell^2)[B_\ell(0) + G_\ell(0) \cdot (1 + h)^{-1} \]

\[ + H_\ell(0) \cdot (1 + h)^{-2}] + O(m^{-3}). \]  

(3.14)

On multiplying the above and (3.13) we obtain:

\[ g_\ell(h) = e^{\ell/2} (1 + h)^{-\ell}[1 + (1/m)E_\ell(h)] \]

\[ + (1/m^2)F_\ell(h)] + O(m^{-3}) \]  

(3.15)

where

\[ E_\ell(h) = [C(\ell) + A_\ell(0)][1 - (1 + h)^{-1}] ; \]  

(3.16)

\[ F_\ell(h) = b_1(\ell) + b_2(\ell) \cdot (1 + h)^{-1} + b_3(\ell) \cdot (1 + h)^{-2}; \]  

(3.17)

\[ b_1(\ell) = B_\ell(0) + D_1(\ell) + A_\ell(0) \cdot C(\ell); \]  

(3.18)

\[ b_2(\ell) = G_\ell(0) + D_2(\ell) - 2A_\ell(0) \cdot C(\ell); \]  

(3.19)

\[ b_3(\ell) = -(b_1(\ell) + b_2(\ell)) \]  

(3.20)

and the various symbols are defined in Lemmas 2 and 3.

Using now (3.15), (3.10) becomes:

\[ g(h) = e^{\ell/2}[1 + (1/m)E_0(h) + c_1(h)(1 + h)^{-1} + c_2(h)(1 + h)^{-2}] \]

\[ + (1/m^2)[F_0(h) + c_4(h)E_1(h)(1 + h)^{-1} \]

and the various symbols are defined in Lemmas 2 and 3. 

Using now (3.15), (3.10) becomes:

\[ g(h) = e^{\ell/2}[1 + (1/m)E_0(h) + c_1(h)(1 + h)^{-1} + c_2(h)(1 + h)^{-2}] \]

\[ + (1/m^2)[F_0(h) + c_4(h)E_1(h)(1 + h)^{-1} \]
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\[ + c_2(h)E_2(h) (1 + h)^{-2} + \sum_{j=1}^{4} d_j(h) \cdot (1 + h)^{-j} \]

\[ + 0(m^3) \]  \hspace{1cm} (3.21)

where \( c_i(h), i = 1,2 \) and \( d_j(h), j = 1, \ldots, 4 \) are available from (3.4) - (3.9), and \( E_2(h), \ell = 0,1,2, F_0(h) \), are available from (3.16) - (3.20). On grouping terms involving \( (1 + h)^{-\ell} \), for \( \ell = 0,1,2 \), from (3.21) we obtain:

\[ g(h) = e^{t_1/2} \{ 1 + (1/m)(\delta_0 + \beta_1(1 + h)^{-1}) \]

\[ + (1/m^2)(\delta_0 + \delta_1(1 + h)^{-1} + \delta_2(1 + h)^{-2}) \]

\[ + 0(m^3) \]  \hspace{1cm} (3.22)

where

\[ \beta_0 = \alpha t_1 - t_2/2 + t_4^2/4p; \]  \hspace{1cm} (3.23)

\[ \beta_1 = t_2/4 - t_4^2/4p; \]  \hspace{1cm} (3.24)

\[ \delta_0 = b_1(0) - (C(1) + A_1(0))t_2/2 + t_3/2 + t_4^2/8; \]  \hspace{1cm} (3.25)

\[ \delta_1 = b_2(0) + (C(1) + A_1(0))\alpha t_1 + [2C(1) + 2A_1(0) + C(2)] \]

\[ + A_2(0) - 4\alpha t_2/4 - t_3/2 - \alpha t_1 t_2/2 + t_4^2/8; \]  \hspace{1cm} (3.26)

\[ \delta_2 = b_3(0) + [\alpha^2 t_4/2 - \alpha C(1) - \alpha A_1(0)]t_4 + [2\alpha - C(2) - A_3(0)] \]

\[ \cdot t_2/4 + t_3/6 + \alpha t_1 t_2/4 + t_4^2/32 \]  \hspace{1cm} (3.27)

where \( b_i(0), i = 1,2,3 \) are given in (3.18) - (3.20), and \( A_k(\ell) \) and \( C(\ell) \) are
given in Lemmas 2 and 3.

3.2. The Main Result.

When $A_1$ is assumed true, letting $q = \sigma^2$ we have:

$$\sigma_{ji} = \sigma^2(1 + \omega_j/m), \ j = 1, \cdots, k; \ i = 1, \cdots, p,$$

$$|q_{E_j}| = \prod_{j=1}^{p} (1 + \omega_j/m).$$

Then, using a logarithmic expansion, we obtain

$$\prod_{j=1}^{k} |q_{E_j}|^m/2 - \Theta_j \alpha = e^{-t_1/2} \left[ 1 + (1/m)^{f_1} + (1/m)^{f_2} \right] + O(m^{-3}) \quad (3.28)$$

where

$$f_1 = t_2/4 - \alpha t_3$$ \quad (3.29)

$$f_2 = \alpha t_2 - t_3/2 + (s^2 t_1 + \alpha t_3 t_2 + t_2/16)/2. \quad (3.30)$$

Thus, the asymptotic expansion of (2.3) can be obtained with the help of (2.6), (3.22) and (3.28). On multiplying these relations we obtain

$$E(\lambda^*)^m h/2 = (1+h)^{-f/2} \left[ 1 + (1/m)[\beta_0^* + \beta_1^*(1+h)^{-.5}] \right]$$

$$+ (1/m^4)[\delta_0^* + \delta_1^*(1 + h)^{-.5} + \delta_2^*(1 + h)^{-.2}] + O(m^{-3}), \quad (3.31)$$

where

$$\beta_0^* = f_1 + \beta_0 = t_2^2/9p - t_3/4;$$ \quad (3.32)

$$\beta_1^* = -\beta_0; \quad (3.33)$$
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\[ \delta_0 = \delta_0 + \beta_0 t_1 + t_2 - \gamma_2 \]
\[ = \left( t_1 - p t_2 \right)^2 / 32 p^2 + \left( t_1 - p t_2 \right) (t_1/2p^2 + \alpha/2p) \]
\[ - \left( t_1 - p t_2 \right)^2 / 3p^2 - \gamma_2; \]
\[ (3.34) \]

\[ \delta_1 = \delta_1 + \beta_1 t_1 = - (t_2 - p t_2)^2 / 16p^2 - (t_1 - p t_2) (1/2p^2 + t_1/p) \]
\[ - (t_1 - p t_2) \alpha/p + (t_1 - p t_2)^2 / 2p^2 ; \]
\[ (3.35) \]

\[ \delta_2 = - (\delta_0 + \delta_1). \]
\[ (3.36) \]

By inverting (3.31) we now obtain the main result:

**Theorem 1.** If alternative \( A_1 \) of (1.8) is assumed true, then for any \( z \):

\[ \text{Pr}(- m \cdot \ln(\lambda^*) \leq z) = \text{Pr}(x_k^2 \leq z) \]
\[ + \left( \beta_0/m \right) \left[ \text{Pr}(x_k^2 \leq z) - \text{Pr}(x_{k+2}^2 \leq z) \right] \]
\[ + \left( 1/m^2 \right) \left[ \delta_0^* \text{Pr}(x_k^2 \leq z) + \delta_1^* \text{Pr}(x_{k+2}^2 \leq z) \right] \]
\[ + \delta_2^* \text{Pr}(x_{k+4}^2 \leq z) \]
\[ + O(m^{-3}) \]

where \( x_n^2 \) denotes a chi-square random variable with \( n \) degrees of freedom.

The above result, for \( k = 1 \) and \( \Omega_1 = -2Q \), reduces to that of Khatri and Srivastava (1974), and for \( \Omega_1 = 0, j = 1, \cdots, k \), it reduces to (2.7).

4. THE DISTRIBUTION OF \( \Lambda^* \) UNDER \( A_2. \)

When \( A_2 \) is assumed true.
\[ |q^E_j| = \prod_{i=1}^{p} (1 - \gamma_{ji}/m)^{-1} \]

where \( \gamma_{ji} \) are the eigenvalues of \( \Gamma_j \), \( j = 1, \ldots, k \), \( i = 1, \ldots, p \). Hence, for large \( m \) we obtain

\[
\begin{align*}
\prod_{i=1}^{k} |q^E_j|^{-\Theta_j m/3 - \Theta_j \alpha} &= e^{-\tau_1/2} \{ 1 + (1/m)f_i \} \\
&\quad + (1/m^2)f_i^2 \} + o(m^{-3})
\end{align*}
\]

(4.1)

where

\[
\tau_1 = \sum_{j=1}^{k} \Theta_j \text{tr} \Gamma_j^2 i = 1, 2, \ldots
\]

(4.2)

\[
f_i = -\tau_2/4 - \alpha \tau_1
\]

(4.3)

\[
f_2 = -\tau_3/6 - \alpha \tau_2/2 + \tau_4^2/32 + \alpha \tau_1 \tau_2/4 + \alpha^2 \tau_1^2/2.
\]

(4.4)

Under \( A_2 \), we also have:

\[
\begin{align*}
\prod_{i=1}^{k} \prod_{j=1}^{p} (1 - z_{ji} y)^{-\Theta_j (1+h) m/2 - \Theta_j \alpha} &= e^{(1+h) \tau_1/2} \{ 1 + (1/m) \\
&\quad \cdot [c^H y + c^H(h)y^2] + (1/m^2) \sum_{j=2}^{4} d_j(h) y^j \} + o(m^{-3}), \quad 0 < y < 1
\end{align*}
\]

(4.5)

where

\[
c^H(h) = c^H = \alpha \tau_1;
\]

(4.6)
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\[ C_{\frac{1}{2}}(h) = (1 + h)r_2/4; \] (4.7)

\[ d_{\frac{1}{2}}(h) = d_{\frac{1}{2}} = (ar_2 + a^2r_1^2)/2; \] (4.8)

\[ d_{\frac{3}{2}}(h) = (1 + h)(r_3/6 + ar_1r_2/4); \] (4.9)

\[ d_{\frac{3}{2}}(h) = (1 + h)r_2^2/32. \] (4.10)

Relation (4.5) is in complete analogy with (3.2) with \( d_{\frac{1}{2}}(h) = 0 \).

Hence, corresponding to (3.10), we now have

\[
g(h) = g_{\frac{1}{2}}(h) + \frac{1}{m} \left[ c_{\frac{1}{2}}(h)g_{\frac{1}{2}}(h) + c_{\frac{3}{2}}(h)g_{\frac{3}{2}}(h) \right]
\]

\[
+ \left( \frac{1}{m^2} \right) \sum_{j=1}^{4} d_{\frac{j}{2}}(h)g_{\frac{j}{2}}(h) + o(m^{-3})
\] (4.11)

where \( g_{\ell}(h), \ell = 0,1,2,3,4 \) is given by (3.11) with \( t_1 \) replaced by \( r_1 \).

Moreover, Lemma 2 holds with no change in \( A_{k}(\ell), B_{k}(\ell), C_{k}(\ell), \) and \( H_{k}(\ell), \) while Lemma 3 holds with \( t_1 \) replaced by \( r_1 \) in (3.13), and also in the expressions for \( C(\ell), D_1(\ell), D_2(\ell) \) and \( D_3(\ell) \), which are now denoted by \( C'(\ell), D_1(\ell), D_2(\ell) \) and \( D_3(\ell) \), respectively.

With the above notation, the asymptotic expansion of \( g_{\ell}(h) \), analogously to (3.15), is

\[
g_{\ell}(h) = e^{r_1/2}(1 + h)^{-\ell}[1 + (1/m)E_{\ell}(h)] + (1/m^2)F_{\ell}(h) + o(m^{-3})
\] (4.12)

where
\[ E_\ell(h) = [C\cdot(\ell) + A_\ell(0)][1 - (1 + h)^{-1}] \]  \hspace{1cm} (4.13)

\[ F_\ell(h) = b_1(\ell) + b_2(\ell) \cdot (1 + h)^{-1} + b_3(\ell) \cdot (1 + h)^{-2}; \]  \hspace{1cm} (4.14)

\[ b_1(\ell) = B_\ell(0) + D_1(\ell) + A_\ell(0) \cdot C\cdot(\ell); \]  \hspace{1cm} (4.15)

\[ b_2(\ell) = G_\ell(0) + D_2(\ell) - 2A_\ell(0) \cdot C\cdot(\ell); \]  \hspace{1cm} (4.16)

\[ b_3(\ell) = -(b_1(\ell) + b_2(\ell)). \]  \hspace{1cm} (4.17)

Hence, in analogy with (3.21) we obtain:

\[
g(h) = e^{\gamma_1/2}[1 + (1/m)E_\ell(h) + c_1(h)(1 + h)^{-1} + c_2(h)(1 + h)^{-2}] \]

\[ + (1/m^2)[F_\ell(h) + c_1(h)E_\ell(h) \cdot (1 + h)^{-1} + c_2(h) \cdot E_\ell(h)] \]

\[ \cdot (1 + h)^{-2} + \sum_{j=1}^{4} d_j(h) \cdot (1 + h)^{-j}] + 0(m^{-3}). \]  \hspace{1cm} (4.18)

Since \( c_1(h) \neq c_1(h), i = 1,2 \) further development does not follow as in Section 3.1. Substitution to (4.18) from (4.6) - (4.10) and (4.13) - (4.17) yields the expression:

\[ g(h) = e^{\gamma_1/2}[1 + (1/m)(\beta_0 + \beta_1(1 + h)^{-1} + (1/m^2)(\delta_0 \]

\[ + \delta_1(1 + h)^{-1} + \delta_2(1 + h)^{-2}) + 0(m^{-3}) \]  \hspace{1cm} (4.19)

where
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\[ \beta_0 = \tau_2^2/4p + \alpha r_1; \]  
\[ \beta_1 = \tau_2/4 - \tau_2^4/4p; \]  
\[ \delta_0 = b_1(0); \]  
\[ \delta_1 = b_2(0) + \alpha r_1[C'(1) + A_1(0)] + \alpha r_2[C'(2) + A_2(0)]/4; \]  
\[ \delta_2 = - (\delta_0 + \delta_1) + \alpha r_2^4/2 + \alpha^2 \tau_2^4/2 \]  
\[ + \tau_3/6 + \alpha r_1 r_2/4 + \tau_2^3/32. \]  

Finally, on multiplying (2.6), (4.1), and (4.19), we obtain moment expression (2.3) under \( A_0 \) as

\[ E(\lambda^* m h/2) = (1 + h)^{-\lambda/2} [1 + (1/m)[\beta_0 + \beta_1(1 + h)^{1}] \]  
\[ + (1/m^2)[\delta_0 + \delta_1(1 + h)^{1} + \delta_2(1 + h)^{2}] \]  
\[ + o(m^{-3}), \]  

where

\[ \tilde{\beta}_0 = \beta_1 + \beta_2 = \tau_2^2/4p - \tau_2/4; \]  
\[ \tilde{\beta}_1 = \beta_1 = - \beta_0; \]  
\[ \tilde{\beta}_2 = \delta_0 + \beta_2 \delta_1 + \delta_2 - \gamma_2 = (\tau_2^4 - p r_2^2)/2p^2 + (\tau_2^4 - p r_2) a/2p \]  
\[ + (\tau_2^2 - p^2 \tau_3)/6p^2 - \gamma_2. \]
\[\bar{\delta}_1 = \delta_1 + \beta_1 f_1 = -\frac{(\tau_1 - p \tau_2)^2}{16p^2} - \frac{(\tau_1 - p \tau_2)(\tau_1 - 2p^2}{2p^2} + \frac{1}{2p^2} + \frac{\alpha}{p}\]  

\[\bar{\delta}_2 = - (\bar{\delta}_0 + \bar{\delta}_1).\]  

By inverting (4.25) we obtain

**Theorem 2.** If alternative \(A_2\) of (1.9) is assumed true, then for any \(z\):

\[\Pr(-m \cdot \kappa \ln(\lambda^*) \leq z) = \Pr(\chi_1^2 \leq z)\]

\[+ \frac{\beta_0}{m} [\Pr(\chi_1^2 \leq z) - \Pr(\chi_{1+2}^2 \leq z)]\]

\[+ \left(1/m^2\right) [\delta_0 \Pr(\chi_1^2 \leq z) + \delta_1 \Pr(\chi_{1+2}^2 \leq z)\]

\[+ \delta_2 \Pr(\chi_{1+4}^2 \leq z)] + O(m^{-2})\]

where \(\beta_0, \beta_1, \delta_0, \delta_1, \delta_2\) are given in (4.25) - (4.29).

As in the previous case of Section 3, when \(k = 1\) and \(\Gamma_1 = 2\) the above result reduces to that of Khatri and Srivastava (1974) and for \(\Gamma_j = 0\), \(j = 1, \ldots, k\), it reduces to (2.7).

5. AN EXAMPLE.

In this section we illustrate the application of our results to Fisher's Iris data (see Fisher (1936)). The data set consists of 50 four-dimensional observations on each of three \((k = 3)\) types of Iris, namely Setosa, Versicolor and Virginica (so \(N_1 = N_2 = N_3 = 50\)). The four \((p = 4)\) measurements are: sepal length (1), sepal width (2), petal length (3) and petal width (4); the data are given in Table 1.

For illustration of the results we consider the hypothesis of (1.7), i.e.
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H: \( \Sigma_1 = \Sigma_2 = \Sigma_3 = \sigma^2 I \)

(5.1)

where \( \Sigma_i \), \( i = 1, 2, 3 \), are the covariance matrices corresponding to the three populations of Iris. To test this hypothesis of multisample sphericity, we first compute the critical value \( z \) such that, if \( H \) is assumed, then \( \Pr(-m \ln \lambda^* \geq z) = .05 \). Then, we compute \(-m \ln \lambda^* \) and compare with the critical value. This can be done using MINITAB. Here we have

\[ k = 3, \ p = 4, \ N = N_1 + N_2 + N_3 = 150, \ N - k = 147, \]

\[ \Theta_j = (N_j - 1)/(N - k) = 1/3, j = 1, 2, 3. \]

With these values, the expressions for \( \alpha, f_0, \) and \( \gamma_2 \) in Lemma 1 give:

\[ \alpha = 2.2227011, \ m = n - 2\alpha = 142.5545977 \]

(5.2)

\[ f = 29, \ \gamma_2 = 356.2283883. \]

(5.3)

To compute the upper 5\% critical value of the test statistic we proceed by trial and error, starting with a computation of the approximation (2.7) for \( z = 41 \) and \( z = 49 \), i.e., two values near the upper 5\% point of a chi-square with 30 degrees of freedom. For these values, using MINITAB, we get \( \Pr(-m \ln \lambda^* \leq 41) = .929545 \) and \( \Pr(-m \ln \lambda^* \leq 49) = .988024 \). Thus, the critical value is between 41 and 49. Continuing in this manner, we find that the critical value is between 42 and 43, then between 42.6 and 42.7, etc., and finally we obtain

\[ \Pr(-m \ln \lambda^* \geq 42.6779) = .05. \]

(5.4)

Thus, the 5\% critical value is 42.6779.

Computation of the test statistic requires the matrix of sums of squares and cross-products \( SS_1, SS_2 \) and \( SS_3 \) corresponding to the three populations. We have:

\[
\begin{array}{cccc}
608.82 & 486.16 & 72.02 & 50.62 \\
486.16 & 704.08 & 52.76 & 45.56 \\
72.02 & 52.76 & 149.22 & 28.82 \\
50.62 & 45.56 & 28.82 & 54.42 \\
\end{array}
\]
MOSCHOPoulos

\[ SS_2 = \begin{bmatrix}
1305.52 & 417.40 & 896.20 & 273.32 \\
417.40 & 482.50 & 405.00 & 201.90 \\
896.20 & 405.00 & 1082.00 & 358.20 \\
273.32 & 201.90 & 358.20 & 191.62 \\
\end{bmatrix} \]

\[ SS_3 = \begin{bmatrix}
1981.28 & 459.44 & 1486.12 & 240.56 \\
459.44 & 509.62 & 349.76 & 233.38 \\
1486.12 & 349.76 & 1492.48 & 239.24 \\
240.56 & 233.38 & 239.24 & 369.62 \\
\end{bmatrix} \]

Further, we compute:

\[ \text{tr}(SS_1) = 1516.54 , \quad |SS_1|^{1/3} = 1078.55 \]  
(5.5)

\[ \text{tr}(SS_2) = 3061.64 , \quad |SS_2|^{1/3} = 2218.25 \]  
(5.6)

\[ \text{tr}(SS_3) = 4222.62 , \quad |SS_3|^{1/3} = 4245.45. \]  
(5.7)

Using the above values, (2.1) gives: \( \ln \lambda^* = -16.4452. \) Since \(-\ln |\lambda^*| = 23.44.34 > 42.6779, \) \( H \) is rejected (as expected) at the 5% level of significance. In the context of the motivating application of the Introduction, if hypothesis (1.6) is rejected, then the F-ratio (1.4) is not valid for testing (1.2). In this case, one may pursue transformations of the data or employ a heteroscedastic model.

The power of the test when alternative A is assumed is computed from Theorem 1 of Section 3. Application of this theorem requires the probabilities

\[ \Pr(\chi^2_{29} \leq 42.6779) = .951258 \]  
(5.8)

\[ \Pr(\chi^2_{31} \leq 42.6779) = .921003 \]  
(5.9)

\[ \Pr(\chi^2_{33} \leq 42.6779) = .879351 \]  
(5.10)
and the coefficients $\beta^*$, $\delta^*$, $\delta^*$, $\delta^*$, given by (3.32), (3.34), (3.35), and (3.36), respectively. The computation of these coefficients from the above relations is trivial and it involves the values of $\alpha$ and $\gamma$ in (5.2) and (5.3) and $t_1$, $t_2$, and $t_3$ of (3.3); the latter depend on the specific alternative $A_1$ in the class (1.8). We specify $A_1$ by the following $\Omega$-matrices.

$$\Omega_1 = \text{diag}(15, -23, 25, 30) \quad \Omega_2 = \text{diag}(10, 20, 17, -15)$$

$$\Omega_3 = \text{diag}(9, 12, -16, 20).$$

Note that, since $m \approx 142$, an entry of $\Omega_i$ equal to 20 (for example), means that the "deviation" from the sphericity hypothesis $H$ is $20/142$ or about 14%. For these $\Omega$'s, we obtain:

$$t_1 = \frac{1}{3} \text{tr}(\Omega_1 + \Omega_2 + \Omega_3) = 34.6666$$

$$t_2 = \frac{1}{3} \text{tr}(\Omega_1^2 + \Omega_2^2 + \Omega_3^2) = 1391.7221$$

$$t_3 = \frac{1}{3} \text{tr}(\Omega_1^3 + \Omega_2^3 + \Omega_3^3) = 16910.6641$$

$$\beta^* = -272.7221 \quad \text{(from (3.32))}$$

$$\delta^* = 35661.5555 \quad \text{(from (3.34))}$$

$$\delta^*_1 = -69515.3125 \quad \text{(from (3.35))}$$

$$\delta^*_2 = 33853.4570 \quad \text{(from (3.36)).}$$

With the above values, from Theorem 1 we compute the power of the test as

$$\text{(Power at } A_1) = 1 - \Pr(-m\bar{n} \lambda^* \leq 42.6779) = .1229.$$
The power of the test when alternative $A_2$ is assumed is computed from Theorem 2 of Section 4. Application of this theorem requires the probabilities in (5.8) — (5.10), and the coefficients $\delta_0$, $\delta_0$, $\delta_1$, $\delta_2$ given by (4.26), (4.28), (4.29), and (4.30), respectively. As in the previous case, these coefficients depend on $\alpha$, $\gamma_1$, $\gamma_2$, and $\tau_3$; the latter depend on the specific alternative $A_2$ in the class (1.9), in particular on the $\Gamma$-matrices, which we choose as

$$\Gamma_1 = \text{diag}(10, 10, -10, -10) \quad \Gamma_2 = \text{diag}(20, 15, 10, -20)$$

$$\Gamma_3 = \text{diag}(10, -10, 10, 15).$$

For these $\Gamma$'s we obtain:

$$\tau_1 = \frac{1}{3} \text{tr}(\Gamma_1 + \Gamma_2 + \Gamma_3) = 16.6666$$

$$\tau_2 = \frac{1}{3} \text{tr}(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2) = 683.3332$$

$$\tau_3 = \frac{1}{3} \text{tr}(\Gamma_1^3 + \Gamma_2^3 + \Gamma_3^3) = 2,916.6665$$

$$\delta_0 = -153.4722 \quad \text{(from (4.26))}$$

$$\delta = 1,030.4961 \quad \text{(from (4.28))}$$

$$\delta_1 = -21,805.5469 \quad \text{(from (4.29))}$$

$$\delta_2 = 11,505.6508 \quad \text{(from (4.30)).}$$

With the above values, from Theorem 2 we compute the power of the test as

$$(\text{Power at } A_2) = 1 - \Pr(-\lambda^* n \leq 42.6779) = .0895.$$
## Table 1. Iris Data

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(1) = sepal length  
(2) = sepal width  
(3) = petal length  
(4) = petal width
ACKNOWLEDGEMENTS.

The author is grateful to a referee for discovering an error in Lemma 2 in an earlier version of the paper. He is also grateful to the editor for valuable comments which have enhanced the article's readability and for suggesting the inclusion of Section 5. The paper was presented by the author at the 17th European Meeting of Statisticians, Thessaloniki, Greece, August 1987.

REFERENCES.


MULTISAMPLE SPHERICITY


Received 10/29/85; Revised 5/24/88.