AMERICAN JOURNAL OF MATHEMATICAL AND MANAGEMENT SCIENCES

ASYMPTOTIC EXPANSIONS OF THE NON–NULL DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERION FOR MULTISAMPLE SPHERICITY

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SYNOPTIC ABSTRACT

Let λ^* be the modified likelihood ratio criterion for testing the multisample sphericity hypothesis H: $\Sigma_j = \sigma^2 I$, $j = 1, \cdots, k$, where Σ_j is the positive definite covariance matrix of the j^{th} normal population. Asymptotic expansions for the distribution of λ^* are derived, under two classes of local alternatives. The results are an extension of those in Khatri and Srivastava (1974) in multisample sphericity.

Key Words and Phrases: Multisample sphericity; likelihood ratio test; asymptotic expansions; local alternatives.

1988, VOL. 8, NOS. 1 & 2, 135-163
0196-6324/88/010135-29 $34.00

1. INTRODUCTION.

Consider the repeated measurements model (1.1) with a fixed number p of responses (collected on the same experimental unit at successive times, or under a variety of experimental conditions)

$$X_{ij} = \mu + \beta_j + e_{ij},\ i = 1,\cdots, k;\ j = 1,\cdots, p \tag{1.1}$$

where μ is the general–level parameter common to all observations, β_j is the measure of the effect specific to the j^{th} condition, and e_{ij} is a random disturbance reflecting both the experimental error and the interaction between the i^{th} unit and the j^{th} response.

Consider the hypothesis of equal response effect

$$H_0\colon \beta_1 = \beta_2 = \cdots = \beta_p \tag{1.2}$$

or, in matrix form, $C\beta = 0$ where C is an orthogonal contrast matrix, for example

$$C = \begin{bmatrix} 1 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & -1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & -1 \end{bmatrix}. \tag{1.3}$$

Let e_i be the vector of the disturbances corresponding to the i^{th} sampling unit, i.e., $e_i = (e_{i1}, \cdots, e_{ip})'$. If the e_i's have the same multivariate normal distribution $N_p(0, \Sigma)$, then H_0 is tested by the well–known Hotelling's T^2 statistic computed from the mean vector and sample covariance matrix of the differences

$$Y_{ij} = X_{ij} - X_{i,j+1}, \ i = 1, \cdots, k; \ j = 1, \cdots, p-1.$$

Denoting by S_y the $(p-1)x(p-1)$ covariance matrix of the Y_{ij} differences, the test statistic is (see Morrison (1976))

$$F = (k - p + 1) \ T^2 \ / \ (k-1)(p-1) \tag{1.4}$$

where

$$T^2 = \bar{Y}' \ S_y^{-1} \ \bar{Y} \tag{1.5}$$

$$\bar{Y} = (\bar{Y}_1, \cdots, \bar{Y}_{p-1})' = (\bar{X}_1 - \bar{X}_2, \cdots, \bar{X}_{p-1} - \bar{X}_p)'.$$

When H_0 is true, F follows an F–distribution with $p - 1$ and $k - p + 1$ degrees of freedom.

However, if the e_i's have different covariance matrices, i.e., $e_i \sim N_p(0, \Sigma_i)$, $i = 1, \cdots, k$, then the above F–ratio may not be valid. An assumption essential to its validity is (see Huynh and Feldt (1970))

$$H_0': C\Sigma_1C' = C\Sigma_2C' = \cdots = C\Sigma_kC' = \sigma^2 I. \tag{1.6}$$

Huynh and Feldt (1970) suggest testing H_0' in two steps: first test the hypothesis H_a: $C\Sigma_jC'$ equal for all j; and second, test the hypothesis H_b: $C\Sigma C' = \sigma^2 I$ where Σ is the pooled dispersion matrix. If H_a and H_b are not rejected, then H_0' is assumed.

The above considerations provide the motivation for a single test of multisample sphericity based on k independent samples from multivariate normal populations. Specifically, let $X_{ji}\{i = 1, \cdots, N_j; \ j = 1, \cdots, k\}$ be k independent samples from normal populations $N_p(\mu_j, \Sigma_j)$ where μ_j is a p x 1 real vector and Σ_j is a positive definite covariance matrix. The hypothesis of multisample sphericity is

$$H: \Sigma_j = \sigma^2 I, \ j = 1, \cdots, k \qquad (1.7)$$

where σ^2 is an unknown scalar. (This is an obvious generalization of the one-sample sphericity hypothesis of Mauchly (1940).)

The hypothesis H is considered in Mendoza (1980) where the null moments of the associated likelihood ratio are given, with a one-term approximation to its null distribution. The approximation consists of a limiting chi-square term, adjusted for faster convergence by the method of Box (1949). Further study of hypothesis H is given by Mathai (1984), who derives various forms of the non-null moments of the modified likelihood ratio test for H, and points out that the exact non-null distribution is complicated and may be obtained in a series of H-functions.

In the one-sample case ($k = 1$), the modified likelihood ratio test has been investigated extensively. The exact non-null distribution is derived in Khatri and Srivastava (1971), and various asymptotic expansions are available. An expansion under fixed alternatives is given in Sugiura (1969) in terms of a normal distribution function and its derivatives, while Khatri and Srivastava (1974) give expansions in terms of chi-squares, under alternatives that are close to the null hypothesis; also see Nagao (1970, 1973).

In this paper we consider the $k \geq 1$ distribution of the modified likelihood ratio test for testing H, under the type of alternatives considered in Khatri and Srivastava (1974) for the $k = 1$ case. In particular we show that (with appropriate modifications of the notation) the results of Khatri and Srivastava can be extended to multisample sphericity. The alternatives are

$$A_1: \Sigma_j = \sigma^2 (I + \frac{\Omega_j}{m}), \ j = 1, \cdots, k, \qquad (1.8)$$

$$A_2: \Sigma_j^{-1} = q(I - \frac{\Gamma_j}{m}), \ j = 1, \cdots, k, \ 0 < q < \infty, \qquad (1.9)$$

where Ω_j and Γ_j are diagonal matrices and

$$m = N_1 + N_2 + \cdots + N_k - k - 2\alpha$$

where α is a constant chosen so as to accelerate convergence of the asymptotic series expansions of the null and non–null distribution; its exact expression is given in Lemma 1 of Section 2.

Under the above alternatives, the distribution of the modified likelihood ratio test for H is expressed in Sections 3 and 4 up to $0(m^{-3})$ in terms of chi–square probability distributions. (This requires a two–term approximation to the null moments, given in Lemma 1 of Section 2.) The final results are given at the ends of Sections 3 and 4. Section 5 contains a comprehensive example.

2. APPROXIMATION TO THE NULL DISTRIBUTION.

The test of H of (1.7) is based on the modified likelihood ratio $(\lambda^*)^{n/2}$, where $n = N - k$, $0 \leq \lambda^* \leq 1$, and rejects H for small values, i.e., smaller than λ_α such that $\Pr(\lambda^* < \lambda_\alpha) = \alpha$ when H is true, where

$$\lambda^* = \left(p^p / \prod_{j=1}^{k} \Theta_j^{\Theta_j p}\right)\left(\prod_{j=1}^{k} |SS_j|^{\Theta_j}\right)(\mathrm{tr}A)^{-p} \tag{2.1}$$

where $N_j - 1 = n_j = \Theta_j n$, $\Sigma_j \Theta_j = 1$, and $A = SS_1 + \cdots + SS_k$, where SS_j is the matrix of sums of squares and cross–products for the j^{th} sample. The h^{th} moment of $(\lambda^*)^{n/2}$ is derived in Mathai (1984) as

$$E(\lambda^*)^{nh/2} = \left[(np)^{np/2} / \prod_{j=1}^{k} n_j^{n_j p/2}\right]^h$$

$$\cdot \prod_{j=1}^{k} \left\{ |q\Sigma_j|^{-n_j/2} \frac{\Gamma_p[n_j(1+h)/2]}{\Gamma_p[n_j/2]} \right\}$$

$$\cdot [\Gamma(nph/2)]^{-1} \int_0^1 y^{np/2-1} (1-y)^{nph/2-1}$$

$$\cdot \prod_{j=1}^{k} \prod_{i=1}^{p} (1 - z_{ji} y)^{-n_j(1+h)/2} \, dy \tag{2.2}$$

where q is an arbitrary positive constant and $z_{ji} = 1 - q^{-1} \sigma_{ji}^{-1}$ where σ_{ji}, i = 1, $\cdots$, p are the eigenvalues of Σ_j, j = 1, $\cdots$, k. Using moment expression (2.2), Mathai obtained alternative representations by noting that the integral part is simply that of an integral representation of Lauricella's F_D hypergeometric function in pk variables. By choosing the arbitrary constant q so that $|z_{ji}| < 1$, the moment is expressible as a convergent multiple series.

On substituting (2/n)h for h, then n = m + 2α, and finally (m/2)h for h (in that order), we obtain

$$E(\lambda^*)^{mh/2} = \phi(h) \prod_{j=1}^{k} |q\Sigma_j|^{-\Theta_j m/2 - \Theta_j \alpha} \cdot g(h) \tag{2.3}$$

where

$$g(h) = \frac{\Gamma[pm(1 + h)/2 + p\alpha]}{\Gamma[pm/2 + p\alpha]\Gamma[pmh/2]}$$

$$\cdot \int_0^1 y^{pm/2+p\alpha-1} (1-y)^{pmh/2-1}$$

$$\cdot \prod_{j=1}^{k} \prod_{i=1}^{p} (1 - z_{ji} y)^{-\Theta_j m(1+h)/2 - \Theta_j \alpha} \, dy, \tag{2.4}$$

and

$$\varphi(h) = \left(p / \prod_{j=1}^{k} \Theta_j^{\Theta_j} \right)^{pmh/2} \cdot \prod_{j=1}^{k} \left\{ \frac{\Gamma_p[\Theta_j m(1 + h)/2 + \Theta_j \alpha]}{\Gamma_p[\Theta_j m/2 + \Theta_j \alpha]} \right\}$$

$$\cdot \frac{\Gamma[pm/2 + p\alpha]}{\Gamma[pm(1 + h)/2 + p\alpha]}. \tag{2.5}$$

When H is true, moment (2.3) reduces to (2.5) which is the null moment. Applying the standard method of Box (1949) (also see Anderson (1984)), we obtain the following result.

Lemma 1. If

$$\alpha = [p(2p^2+3p-1)\sum_{j=1}^{k}(1/12\Theta_j)-(1/3p)]/[kp(p+1)-2]$$

then

$$\varphi(h) = (1+h)^{-f/2}\{1+\gamma_2[(1+h)^{-2}-1]/m^2+0(m^{-3})\} \tag{2.6}$$

where

$$f = kp(p+1)/2-1$$

$$\gamma_2 = f\alpha^2 + S_1\alpha + S_0$$

$$S_0 = [p(p-1)(5p^2-p-2)/48]\sum_{j=1}^{k}\Theta_j^{-2}$$

$$S_1 = -[p(2p^2+3p-1)/12]\sum_{j=1}^{k}\Theta_j^{-1} + (1/3p). \ \square$$

It follows from (2.6) that

$$P[-m\ell n\,\lambda^* \le z] = P[\chi_f^2 \le z]$$
$$+(\gamma_2/m^2) \cdot [P[\chi_{f+4}^2 \le z] - P[\chi_f^2 \le z]] + 0(m^{-3}). \tag{2.7}$$

To assess the accuracy of the above approximation, exact percentage points for the distribution of λ^* are required; these are presently unavailable. For $k = 1$, the above reduces to the well–known approximation in the one–sample case; see Anderson (1984, p. 432).

3. THE DISTRIBUTION OF λ^* UNDER A_1.

Consider moment expression (2.3). since the expansion of $\phi(h)$ is available from Lemma 1, we need to expand the other factors. We begin by expanding g(h) under A_1.

Since q is arbitrary, it may be chosen so that $|z_{ji}| < 1$, i = 1, ⋯,

$p, j = 1, \cdots, k$ and hence the product involving z_{ji} in (2.4) may be expanded using a logarithmic expansion. The choice $q = \sigma^{-2}$, under A_1 gives

$$z_{ji} = 1 - q^{-1}\sigma_{ji}^{-1} = 1 - (1 + \omega_{ji}/m)$$

$$= \omega_{ji}/m - \omega_{ji}^2/m^2 + \omega_{ji}^3/m^3 + \ldots \tag{3.1}$$

where ω_{ji} are the diagonal elements of Ω_j, $i = 1, \cdots, p$, $j = 1, \cdots, k$. Then for $\min(\omega_{ji}) > -m/2$ it is $|z_{ji}| < 1$. Hence, for $0 < y < 1$ we obtain

$$R(y) = \prod_{j=1}^{k}\prod_{i=1}^{p}(1 - z_{ji}y)^{-\Theta_j m(1+h)/2 - \Theta_j\alpha} = \exp\{\ell n R(y)\}$$

$$e^{(1+h)t_1 y/2}\{1 + (1/m)\cdot(c_1(h)y + c_2(h)y^2)$$

$$+ (1/m^2)\sum_{j=1}^{4} d_j(h)y^j + 0(m^{-3})\} \tag{3.2}$$

where

$$t_i = \sum_{j=1}^{k}\Theta_j\, tr(\Omega_j^i),\ i = 1, 2, \cdots \tag{3.3}$$

$$c_1(h) = \alpha t_1 - t_2(1 + h)/2 \tag{3.4}$$

$$c_2(h) = t_2(1 + h)/4 \tag{3.5}$$

$$d_1(h) = -\alpha t_2 + (1 + h)\, t_3/2 \tag{3.6}$$

$$d_2(h) = \alpha t_2/2 + \alpha^2 t_1^2/2 - (t_3 + \alpha t_1 t_2)\cdot(1 + h)/2$$

$$+ t_2^2(1 + h)^2/8 \tag{3.7}$$

$$d_3(h) = (t_3/6 + \alpha t_1 t_2/4)(1 + h) - t_2^2(1 + h)^2/8 \tag{3.8}$$

$$d_4(h) = t_2^2(1 + h)^2/32. \tag{3.9}$$

Using (3.2), (2.4) becomes

$$g(h) = \frac{\Gamma[pm(1 + h)/2 + p\alpha]}{\Gamma[pm/2 + p\alpha]\,\Gamma[pmh/2]}$$

$$\cdot \left\{ \int_0^1 y^{pm/2+p\alpha-1} (1-y)^{pmh/2-1} e^{(1+h)t_1 y/2}\, dy \right.$$

$$+ (1/m)[c_1(h) \int_0^1 y^{pm/2+p\alpha} (1-y)^{pmh/2-1} e^{(1+h)t_1 y/2}\, dy$$

$$+ c_2(h) \int_0^1 y^{pm/2+p\alpha+1} (1-y)^{pmh/2-1} e^{(1+h)t_1 y/2}\, dy]$$

$$+ (1/m^2) \sum_{j=1}^{4} d_j(h) \int_0^1 y^{pm/2+p\alpha+j-1} (1-y)^{pmh/2-1}$$

$$\left. \cdot\, e^{(1+h)t_1 y/2}\, dy + 0(m^{-3}) \right\}$$

$$= g_0(h) + (1/m)[c_1(h)\, g_1(h) + c_2(h)\, g_2(h)]$$

$$+ (1/m^2) \sum_{j=1}^{4} d_j(h)\, g_j(h) + 0(m^{-3}) \tag{3.10}$$

where for $\ell = 0, 1, 2, 3, 4$, $g_\ell(h)$ is given by

$$g_\ell(h) = \frac{(pm/2 + p\alpha)_\ell}{[pm(1 + h)/2 + p\alpha]_\ell}.$$

$$\sum_{k=0}^{\infty} \frac{(pm/2 + p\alpha + \ell)_k \; ((1 + h)t_1/2)^k}{[pm(1 + h)/2 + p\alpha + \ell]_k \; k!} \tag{3.11}$$

with $(a)_k = a(a + 1) \cdots (a + k - 1)$, $(a)_0 = 1$. Equality (3.10) follows from the following integral and series representation of an ${}_1F_1(a;c;z)$ hypergeometric function; see Erdelyi, Magnus, Oberhettinger, and Tricomi (1953), p. 255:

$$\frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 y^{a-1} (1-y)^{c-a-1} e^{zy}\, dy$$

$$= {}_1F_1(a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k \; z^k}{(c)_k \; k!}, \; \mathrm{Re}(c) > \mathrm{Re}(a) > 0$$

and z is free of y. In obtaining (3.10), the above series representation was used with $a = pm/2 + p\alpha + j$, $c = pmh/2 + p\alpha + j$, $z = (1 + h)t_1/2$ and h such that the conditions are satisfied. An asymptotic expansion of $g_\ell(h)$ up to the order $0(m^{-3})$ will be obtained next.

3.1. Asymptotic Expansion of $g_\ell(h)$.

Lemma 2. For $\ell = 0,1,2,3,4$ and large m we have:

$$\frac{(pm/2 + p\alpha + \ell)_k}{[pm(1 + h)/2 + p\alpha + \ell]_k} = (1 + h)^{-k} \{1 + (1/m)$$

$$\cdot A_k(\ell)[1 - (1 + h)^{-1}] + (1/m^2)[B_k(\ell) + G_k(\ell) \cdot (1 + h)^{-1}$$

$$+ H_k(\ell) \cdot (1 + h)^{-2}]\} + 0(m^{-3}), \tag{3.12}$$

where $\gamma_\ell = 2p\alpha + 2\ell$,

$$A_k(\ell) = k(\gamma_\ell + k - 1)/p;$$

$$B_k(\ell) = (1/6p^2)\{3k^4 + (6\gamma_\ell - 10)k^3 + (3\gamma_\ell^2 - 12\gamma_\ell + 9)k^2$$

$$- (3\gamma_\ell^2 - 6\gamma_\ell + 2)k\};$$

$$G_k(\ell) = -\{k^4 + 2(\gamma_\ell - 1)k^3 + (\gamma_\ell - 1)^2k^2\}/p^2;$$

$$H_k(\ell) = -(B_k(\ell) + G_k(\ell)).$$

Proof: The proof is based on the following well-known asymptotic expansion of the ratio of two gamma functions; see for example Luke (1969), p. 33:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}[1 + \frac{(a-b)(a+b-1)}{2z} + \frac{(a-b)(a-b-1)}{24z^2}$$

$$\cdot \{3(a+b-1)^2 - a + b - 1\}][1 + 0(z^{-3})]$$

where a and b are bounded constants, $b - a > 0$ and $z \to \infty$. Using the above, we have:

$$\frac{(pm/2 + p\alpha + \ell)_k}{[pm(1+h)/2 + p\alpha + \ell]_k} = \frac{\Gamma[pm/2 + p\alpha + \ell + k]}{\Gamma[pm/2 + p\alpha + \ell]}$$

$$\cdot \frac{\Gamma[pm(1+h)/2 + p\alpha + \ell]}{\Gamma[pm(1+h)/2 + p\alpha + \ell + k]}$$

$$= (pm/2)^k[1 + \frac{k(2p\alpha + 2\ell + k - 1)}{pm} + \frac{k(k-1)}{6p^2m^2}$$

$$\cdot \{3(2p\alpha + 2\ell + k - 1)^2 - k - 1\}][1 + 0(m^{-3})]$$

$$= (pm/2)^{-k}(1+h)^{-k}[1 - \frac{k(2p\alpha + 2\ell + k - 1)}{pm(1+h)}$$

$$+ \frac{k(k+1)}{6p^2m^2(1+h)^2} \cdot \{3(2p\alpha + 2\ell + k - 1)^2 + k - 1\}]$$

$$\cdot\ [1 + 0(m^{-3})],$$

and the lemma follows after some algebra. □

Lemma 3. For $\ell = 0,1,2,3,4$ and large m, we have:

$$\sum_{k=0}^{\infty} \frac{(pm/2 + p\alpha + \ell)_k[(1 + h)t_1/2]^k}{[pm(1 + h)/2 + p\alpha + \ell]_k\, k!}$$

$$= e^{t_1/2} \cdot \{1 + (1/m) \cdot C(\ell)[1 - (1 + h)^{-1}]$$

$$+ (1/m^2)[D_1(\ell) + D_2(\ell) \cdot (1 + h)^{-1}$$

$$+ D_3(\ell) \cdot (1 + h)^{-2}]\} + 0(m^{-3}), \tag{3.13}$$

where

$$C(\ell) = [(t_1/2)^2 + \gamma_\ell\, t_1/2]/p;$$

$$D_1(\ell) = (1/6p^2)\ \{3(t_1/2)^4 + (6\gamma_\ell + 8)(t_1/2)^3$$

$$+ (3\gamma_\ell^2 + 6\gamma_\ell)(t_1/2)^2\};$$

$$D_2(\ell) = (-1/p^2)\ \{(t_1/2)^4 + (2\gamma_\ell + 4)(t_1/2)^3$$

$$+ (\gamma_\ell^2 + 4\gamma_\ell + 2)(t_1/2)^2 + \gamma_\ell^2\, t_1/2$$

$$D_3(\ell) = -(D_1(\ell) + D_2(\ell))$$

and $\gamma_\ell = 2p\alpha + 2\ell$.

Proof: On substituting (3.12) in the left–hand side of (3.13), it becomes:

$$\sum_{k=0}^{\infty} \frac{(pm/2 + p\alpha + \ell)_k[(1 + h)t_1/2]^k}{[pm(1 + h)/2 + p\alpha + \ell]_k\, k!}$$

$$= e^{t_1/2}\Bigg\{1 + (1/m)[1 - (1 + h)^{-1}] \sum_{k=0}^{\infty} A_k(\ell)\, \frac{(t_1/2)^k}{k!}$$

$$+ (1/m^2)\Bigg[\sum_{k=0}^{\infty} B_k(\ell)\frac{(t_1/2)^k}{k!} + (1 + h)^{-1} \sum_{k=0}^{\infty} G_k(\ell)\, \frac{(t_1/2)^k}{k!}$$

$$+ (1 + h)^{-2} \sum_{k=0}^{\infty} H_k(\ell)\, \frac{(t_1/2)^k}{k!}\Bigg]\Bigg\} + 0(m^{-3}).$$

The lemma now follows by summing the above series using the formulas

$$k^2 = k(k-1) + k;$$

$$k^3 = k(k-1)(k-2) + 3k(k-1) + k;$$

$$k^4 = k(k-1)(k-2)(k-3) + 6k(k-1)(k-2) + 7k(k-1) + k;$$

$$\sum_{k=0}^{\infty} k(t_1/2)^k/k! = (t_1/2)e^{t_1/2};$$

$$\sum_{k=0}^{\infty} k^2(t_1/2)^k/k! = [(t_1/2)^2 + t_1/2]e^{t_1/2};$$

$$\sum_{k=0}^{\infty} k^3(t_1/2)^k/k! = [(t_1/2)^3 + 3(t_1/2)^2 + (t_1/2)]e^{t_1/2};$$

$$\sum_{k=0}^{\infty} k^4(t_1/2)^k/k! = [t_1/2)^4 + 6(t_1/2)^3 + 7(t_1/2)^2 + t_1/2]e^{t_1/2}. \ \square$$

With the help of Lemmas 2 and 3, we can now obtain the asymptotic expansion of $g_\ell(h)$ in (3.11). Application of (3.12) with $\ell = 0$ and $k = \ell$ gives:

$$\frac{(pm/2 + p\alpha)_\ell}{[pm(1 + h)/2 + p\alpha]_\ell} = (1 + h)^{-\ell}\{1 + (1/m)A_\ell(0)$$

$$\cdot [1 - (1 + h)^{-1}] + (1/m^2)[B_\ell(0) + G_\ell(0) \cdot (1 + h)^{-1}$$

$$+ H_\ell(0) \cdot (1 + h)^{-2}]\} + 0(m^{-3}). \tag{3.14}$$

On multiplying the above and (3.13) we obtain:

$$g_\ell(h) = e^{t_1/2} (1 + h)^{-\ell}[1 + (1/m)E_\ell(h)$$

$$+ (1/m^2)F_\ell(h)] + 0(m^{-3}) \tag{3.15}$$

where

$$E_\ell(h) = [C(\ell) + A_\ell(0)][1 - (1 + h)^{-1}]; \tag{3.16}$$

$$F_\ell(h) = b_1(\ell) + b_2(\ell)\cdot(1+h)^{-1} + b_3(\ell)\cdot(1+h)^{-2}; \tag{3.17}$$

$$b_1(\ell) = B_\ell(0) + D_1(\ell) + A_\ell(0) \cdot C(\ell); \tag{3.18}$$

$$b_2(\ell) = G_\ell(0) + D_2(\ell) - 2A_\ell(0) \cdot C(\ell); \tag{3.19}$$

$$b_3(\ell) = -(b_1(\ell) + b_2(\ell)) \tag{3.20}$$

and the various symbols are defined in Lemmas 2 and 3.

Using now (3.15), (3.10) becomes:

$$g(h) = e^{t_1/2}\{1 + (1/m)[E_0(h) + c_1(h)(1 + h)^{-1} + c_2(h)(1 + h)^{-2}]$$

$$+ (1/m^2)[F_0(h) + c_1(h)E_1(h)\cdot(1 + h)^{-1}$$

$$+ c_2(h)E_2(h)\,(1+h)^{-2} + \sum_{j=1}^{4} d_j(h)\cdot(1+h)^{-j}]\}$$

$$+ 0(m^{-3}) \tag{3.21}$$

where $c_i(h)$, $i = 1,2$ and $d_j(h)$, $j = 1, \cdots, 4$ are available from (3.4) – (3.9), and $E_\ell(h)$, $\ell = 0,1,2$, $F_0(h)$, are available from (3.16) – (3.20). On grouping terms involving $(1 + h)^{-\ell}$, for $\ell = 0,1,2$, from (3.21) we obtain:

$$g(h) = e^{t_1/2}\{1 + (1/m)(\beta_0 + \beta_1(1+h)^{-1})$$

$$+ (1/m^2)(\delta_0 + \delta_1(1+h)^{-1} + \delta_2(1+h)^{-2})\}$$

$$+ 0(m^{-3}) \tag{3.22}$$

where

$$\beta_0 = \alpha t_1 - t_2/2 + t_1^2/4p; \tag{3.23}$$

$$\beta_1 = t_2/4 - t_1^2/4p; \tag{3.24}$$

$$\delta_0 = b_1(0) - (C(1) + A_1(0))t_2/2 + t_3/2 + t_2^2/8; \tag{3.25}$$

$$\delta_1 = b_2(0) + (C(1) + A_1(0))\alpha t_1 + [2C(1) + 2A_1(0) + C(2)$$

$$+ A_2(0) - 4\alpha]t_2/4 - t_3/2 - \alpha t_1 t_2/2 - t_2^2/8; \tag{3.26}$$

$$\delta_2 = b_3(0) + [\alpha^2 t_1/2 - \alpha C(1) - \alpha A_1(0)]t_1 + [2\alpha - C(2) - A_2(0)]$$

$$\cdot\, t_2/4 + t_3/6 + \alpha t_1 t_2/4 + t_2^2/32 \tag{3.27}$$

where $b_i(0)$, $i = 1,2,3$ are given in (3.18) – (3.20), and $A_k(\ell)$ and $C(\ell)$ are given in Lemmas 2 and 3.

3.2. The Main Result.

When A_1 is assumed true, letting $q = \sigma^{-2}$ we have:

$$\sigma_{ji} = \sigma^2(1 + \omega_{ji}/m),\ j = 1, \cdots, k;\ i = 1, \cdots, p,$$

$$|q\Sigma_j| = \prod_{i=1}^{p} (1 + \omega_{ji}/m).$$

Then, using a logarithmic expansion, we obtain

$$\prod_{j=1}^{k} |q\Sigma_j|^{-\Theta_j m/2 - \Theta_j \alpha} = e^{-t_1/2} [1 + (1/m)f_1 + (1/m^2)f_2] + 0(m^{-3}) \tag{3.28}$$

where

$$f_1 = t_2/4 - \alpha t_1 \tag{3.29}$$

$$f_2 = \alpha t_2/2 - t_3/6 + (a^2 t_1^2 - \alpha t_1 t_2/2 + t_2^2/16)/2. \tag{3.30}$$

Thus, the asymptotic expansion of (2.3) can be obtained with the help of (2.6), (3.22) and (3.28). On multiplying these relations we obtain

$$E(\lambda^*)^{mh/2} = (1+h)^{-f/2}\{1 + (1/m)[\beta_0^* + \beta_1^*(1+h)^{-1}] + (1/m^2)[\delta_0^* + \delta_1^*(1 + h)^{-1} + \delta_2^*(1 + h)^{-2}]\} + 0(m^{-3}), \tag{3.31}$$

where

$$\beta_0^* = f_1 + \beta_0 = t_1^2/4p - t_2/4; \tag{3.32}$$

$$\beta_1^* = -\beta_0^*; \tag{3.33}$$

$$\overset{*}{\delta_0} = \delta_0 + \beta_0 f_1 + f_2 - \gamma_2$$

$$= (t_1^2 - pt_2)^2/32p^2 + (t_1^2 - pt_2)(t_1/2p^2 + \alpha/2p)$$

$$- (t_1^3 - p^2 t_3)/3p^2 - \gamma_2; \tag{3.34}$$

$$\overset{*}{\delta_1} = \delta_1 + \beta_1 f_1 = -(t^2 - pt_2)^2/16p^2 - (t_1^2 - pt_2)(1/2p^2 + t_1/p^2)$$
$$- (t_1^2 - pt_2)\ \alpha/p + (t_1^3 - p^2 t_3)/2p^2; \tag{3.35}$$

$$\overset{*}{\delta_2} = -(\overset{*}{\delta_0} + \overset{*}{\delta_1}). \tag{3.36}$$

By inverting (3.31) we now obtain the main result:

Theorem 1. If alternative A_1 of (1.8) is assumed true, then for any z:

$$\Pr(-m \cdot \ell n(\lambda^*) \le z) = \Pr(\chi_f^2 \le z)$$

$$+ (\overset{*}{\beta_0}/m)[\Pr(\chi_f^2 \le z) - \Pr(\chi_{f+2}^2 \le z)]$$

$$+ (1/m^2)[\overset{*}{\delta_0} \Pr(\chi_f^2 \le z) + \overset{*}{\delta_1} \Pr(\chi_{f+2}^2 \le z)$$

$$+ \overset{*}{\delta_2} \Pr(\chi_{f+4}^2 \le z)] + 0(m^{-3})$$

where χ_n^2 denotes a chi–square random variable with n degrees of freedom.

The above result, for k = 1 and $\Omega_1 = -2Q$, reduces to that of Khatri and Srivastava (1974), and for $\Omega_j = 0$, j = 1, $\cdots$, k, it reduces to (2.7).

4. THE DISTRIBUTION OF λ^* UNDER A_2.

When A_2 is assumed true,

$$|q\Sigma_j| = \prod_{i=1}^{p} (1 - \gamma_{ji}/m)^{-1}$$

where γ_{ji} are the eigenvalues of Γ_j, j = 1, $\cdots$, k, i = 1, $\cdots$, p. Hence, for large m we obtain

$$\prod_{j=1}^{k} |q\Sigma_j|^{-\Theta_j m/3 - \Theta_j \alpha} = e^{-\tau_1/2}\{1 + (1/m)f_1'$$

$$+ (1/m^2)f_2'\} + 0(m^{-3}) \tag{4.1}$$

where

$$\tau_i = \sum_{j=1}^{k} \Theta_j \operatorname{tr}\Gamma_j^i \quad i = 1, 2, \cdots \tag{4.2}$$

$$f_1' = -\tau_2/4 - \alpha\tau_1 \tag{4.3}$$

$$f_2' = -\tau_3/6 - \alpha\tau_2/2 + \tau_2^2/32 + \alpha\tau_1\tau_2/4 + \alpha^2\tau_1^2/2. \tag{4.4}$$

Under A_2, we also have:

$$\prod_{j=1}^{k} \prod_{i=1}^{p} (1 - z_{ji}y)^{-\Theta_j(1+h)m/2 - \Theta_j\alpha} = e^{(1+h)\tau_1/2}\{1 + (1/m)$$

$$\cdot [c_1'y + c_2'(h)y^2] + (1/m^2) \sum_{j=2}^{4} d_j'(h)y^j$$

$$+ 0(m^{-3}), \quad 0 < y < 1 \tag{4.5}$$

where

$$C_1'(h) = c_1' = \alpha\tau_1; \tag{4.6}$$

$$C_2'(h) = (1 + h)\tau_2/4; \tag{4.7}$$

$$d_2'(h) = d_2' = (\alpha\tau_2 + \alpha^2\tau_1^2)/2; \tag{4.8}$$

$$d_3'(h) = (1 + h)(\tau_3/6 + \alpha\tau_1\tau_2/4); \tag{4.9}$$

$$d_4'(h) = (1 + h)^2\tau_2^2/32. \tag{4.10}$$

Relation (4.5) is in complete analogy with (3.2) with $d_1'(h) = 0$. Hence, corresponding to (3.10), we now have

$$g(h) = g_0'(h) + (1/m)[c_1'(h)g_1'(h) + c_2'(h)g_2'(h)]$$

$$+ (1/m^2) \sum_{j=1}^{4} d_j'(h)g_j'(h) + 0(m^{-3}) \tag{4.11}$$

where $g_\ell'(h)$, $\ell = 0,1,2,3,4$ is given by (3.11) with t_1 replaced by τ_1. Moreover, Lemma 2 holds with no change in $A_k(\ell)$, $B_k(\ell)$, $G_k(\ell)$, and $H_k(\ell)$, while Lemma 3 holds with t_1 replaced by τ_1 in (3.13), and also in the expressions for $C(\ell)$, $D_1(\ell)$, $D_2(\ell)$ and $D_3(\ell)$, which are now denoted by $C'(\ell)$, $D_1'(\ell)$, $D_2'(\ell)$ and $D_3'(\ell)$, respectively.

With the above notation, the asymptotic expansion of $g_\ell'(h)$, analogously to (3.15), is

$$g_\ell'(h) = e^{\tau_1/2}(1 + h)^{-\ell}[1 + (1/m)E_\ell'(h)$$

$$+ (1/m^2)F_\ell'(h)] + 0(m^{-3}) \tag{4.12}$$

where

$$E'_\ell(h) = [C'(\ell) + A_\ell(0)][1-(1+h)^{-1}]; \tag{4.13}$$

$$F'_\ell(h) = b'_1(\ell) + b'_2(\ell)\cdot(1+h)^{-1}$$

$$+ b'_3(\ell)\cdot(1+h)^{-2}; \tag{4.14}$$

$$b'_1(\ell) = B_\ell(0) + D'_1(\ell) + A_\ell(0)\cdot C'(\ell); \tag{4.15}$$

$$b'_2(\ell) = G_\ell(0) + D'_2(\ell) - 2A_\ell(0)\cdot C'(\ell); \tag{4.16}$$

$$b'_3(\ell) = -(b'_1(\ell) + b'_2(\ell)). \tag{4.17}$$

Hence, in analogy with (3.21) we obtain:

$$g(h) = e^{\tau_1/2}\{1 + (1/m)[E'_0(h) + c'_1(h)(1+h)^{-1} + c'_2(h)(1+h)^{-2}]$$

$$+ (1/m^2)[F'_0(h) + c'_1(h)E'_1(h)\cdot(1+h)^{-1} + c'_2(h)\cdot E'_2(h)$$

$$\cdot(1+h)^{-2} + \sum_{j=1}^{4} d'_j(h)\cdot(1+h)^{-j}]\} + 0(m^{-3}). \tag{4.18}$$

Since $c'_i(h) \neq c_i(h)$, $i = 1,2$ further development does not follow as in Section 3.1. Substitution to (4.18) from (4.6) – (4.10) and (4.13) – (4.17) yields the expression

$$g(h) = e^{\tau_1/2}\{1 + (1/m)(\beta'_0 + \beta'_1(1+h)^{-1}) + (1/m^2)(\delta'_0$$

$$+ \delta'_1(1+h)^{-1} + \delta'_2(1+h)^{-2})\} + 0(m^{-3}) \tag{4.19}$$

where

$$\beta_0' = \tau_1^2/4p + \alpha\tau_1; \tag{4.20}$$

$$\beta_1' = \tau_2/4 - \tau_1^2/4p; \tag{4.21}$$

$$\delta_0' = b_1'(0); \tag{4.22}$$

$$\delta_1' = b_2'(0)+\alpha\tau_1[C'(1)+A_1(0)]+\tau_2[C'(2)+A_2(0)]/4; \tag{4.23}$$

$$\delta_2' = -(\delta_0' + \delta_1') + \alpha\tau_2/2 + \alpha^2\tau_1^2/2$$

$$+ \tau_3/6 + \alpha\tau_1\tau_2/4 + \tau_2^2/32. \tag{4.24}$$

Finally, on multiplying (2.6), (4.1), and (4.19), we obtain moment expression (2.3) under A_2 as

$$E(\lambda^*)^{mh/2} = (1 + h)^{-f/2}\{1 + (1/m)[\tilde\beta_0 + \tilde\beta_1(1 + h)^{-1}]$$

$$+ (1/m^2)[\tilde\delta_0 + \tilde\delta_1(1 + h)^{-1} + \tilde\delta_2(1 + h)^{-2}]\}$$

$$+ 0(m^{-3}), \tag{4.25}$$

where

$$\tilde\beta_0 = f_1' + \beta_0' = \tau_1^2/4p - \tau_2/4; \tag{4.26}$$

$$\tilde\beta_1 = \beta_1' = -\tilde\beta_0; \tag{4.27}$$

$$\tilde\delta_0 = \delta_0' + \beta_0'f_1' + f_2' - \gamma_2 = (\tau_1^2 - p\tau_2)^2/32p^2 + (\tau_1^2 - p\tau_2)\alpha/2p$$

$$+ (\tau_1^3 - p^2\tau_3)/6p^2 - \gamma_2 \tag{4.28}$$

$$\tilde{\delta}_1 = \delta_1' + \beta_1' f_1' = -(\tau_1^2 - p\tau_2)^2/16p^2 - (\tau_1^2 - p\tau_2)(\tau_1/2p^2$$

$$+ 1/2p^2 + \alpha/p) \tag{4.29}$$

$$\tilde{\delta}_2 = -(\tilde{\delta}_0 + \tilde{\delta}_1). \tag{4.30}$$

By inverting (4.25) we obtain

Theorem 2. If alternative A_2 of (1.9) is assumed true, then for any z:

$$\Pr(-m \cdot \ell n(\lambda^*) \leq z) = \Pr(\chi_f^2 \leq z)$$

$$+ (\tilde{\beta}_0/m)[\Pr(\chi_f^2 \leq z) - \Pr(\chi_{f+2}^2 \leq z)]$$

$$+ (1/m^2)[\tilde{\delta}_0 \Pr(\chi_f^2 \leq z) + \tilde{\delta}_1 \Pr(\chi_{f+2}^2 \leq z)$$

$$+ \tilde{\delta}_2 \Pr(\chi_{f+4}^2 \leq z)] + 0(m^{-3})$$

where $\tilde{\beta}_0$, $\tilde{\beta}_1$, $\tilde{\delta}_0$, $\tilde{\delta}_1$, $\tilde{\delta}_2$, are given in (4.25) – (4.29).

As in the previous case of Section 3, when k = 1 and $\Gamma_1 = 2P$ the above result reduces to that of Khatri and Srivastava (1974) and for $\Gamma_j = 0$, $j = 1, \cdots, k$, it reduces to (2.7).

5. AN EXAMPLE.

In this section we illustrate the application of our results to Fisher's Iris data (see Fisher (1936)). The data set consists of 50 four–dimensional observations on each of three (k = 3) types of Iris, namely Setosa, Versicolor and Virginica (so $N_1 = N_2 = N_3 = 50$). The four (p = 4) measurements are: sepal length (1), sepal width (2), petal length (3) and petal width (4); the data are given in Table 1.

For illustration of the results we consider the hypothesis of (1.7), i.e.

$$H:\ \Sigma_1 = \Sigma_2 = \Sigma_3 = \sigma^2 I \tag{5.1}$$

where Σ_j, j = 1,2,3, are the covariance matrices corresponding to the three populations of Iris. To test this hypothesis of multisample sphericity, we first compute the critical value z such that, if H is assumed, then $\Pr(-m\ell n\ \lambda^* \geq z) = .05$. Then, we compute $-m\ell n\ \lambda^*$ and compare with the critical value. This can be done using MINITAB. Here we have

$k = 3,\ p = 4,\ N = N_1 + N_2 + N_3 = 150,\ N - k = 147,$

$\Theta_j = (N_j - 1)/(N - k) = 1/3, j = 1,2,3.$

With these values, the expressions for α, f_0, and γ_2 in Lemma 1 give:

$$\alpha = 2.2227011,\ m = n - 2\alpha = 142.5545977 \tag{5.2}$$

$$f = 29,\ \gamma_2 = 356.2283883. \tag{5.3}$$

To compute the upper 5% critical value of the test statistic we proceed by trial and error, starting with a computation of the approximation (2.7) for z = 41 and z = 49, i.e., two values near the upper 5% point of a chi–square with 30 degrees of freedom. For these values, using MINITAB, we get $\Pr(-m\ell n\ \lambda^* \leq 41) = .929545$ and $\Pr(-m\ell n\ \lambda^* \leq 49) = .988024$. Thus, the critical value is between 41 and 49. Continuing in this manner, we find that the critical value is between 42 and 43, then between 42.6 and 42.7, etc., and finally we obtain

$$\Pr(-m\ell n\ \lambda^* \geq 42.6779) = .05. \tag{5.4}$$

Thus, the 5% critical value is 42.6779.

Computation of the test statistic requires the matrices of sums of squares and cross–products SS_1, SS_2 and SS_3 corresponding to the three populations. We have:

$$SS_1 = \begin{vmatrix} 608.82 & 486.16 & 72.02 & 50.62 \\ 486.16 & 704.08 & 52.76 & 45.56 \\ 72.02 & 52.76 & 149.22 & 28.82 \\ 50.62 & 45.56 & 28.82 & 54.42 \end{vmatrix}$$

$$SS_2 = \begin{vmatrix} 1305.52 & 417.40 & 896.20 & 273.32 \\ 417.40 & 482.50 & 405.00 & 201.90 \\ 896.20 & 405.00 & 1082.00 & 358.20 \\ 273.32 & 201.90 & 358.20 & 191.62 \end{vmatrix}$$

$$SS_3 = \begin{vmatrix} 1981.28 & 459.44 & 1486.12 & 240.56 \\ 459.44 & 509.62 & 349.76 & 233.38 \\ 1486.12 & 349.76 & 1492.48 & 239.24 \\ 240.56 & 233.38 & 239.24 & 369.62 \end{vmatrix}$$

Further, we compute:

$$\mathrm{tr}(SS_1) = 1{,}516.54 \ , \ |SS_1|^{1/3} = 1{,}078.55 \tag{5.5}$$

$$\mathrm{tr}(SS_2) = 3{,}061.64 \ , \ |SS_2|^{1/3} = 2{,}218.25 \tag{5.6}$$

$$\mathrm{tr}(SS_3) = 4{,}222.62 \ , \ |SS_3|^{1/3} = 4{,}245.45. \tag{5.7}$$

Using the above values, (2.1) gives: $\ell n\ \lambda^* = -16.4452$. Since $-m\ell n\ \lambda^* = 2{,}344.34 > 42.6779$, H is rejected (as expected) at the 5% level of significance. In the context of the motivating application of the Introduction, if hypothesis (1.6) is rejected, then the F–ratio (1.4) is not valid for testing (1.2). In this case, one may pursue transformations of the data or employ a heteroscedastic model.

The power of the test when alternative A_1 is assumed is computed from Theorem 1 of Section 3. Application of this theorem requires the probabilities

$$\Pr(\chi^2_{29} \le 42.6779) = .951258 \tag{5.8}$$

$$\Pr(\chi^2_{31} \le 42.6779) = .921003 \tag{5.9}$$

$$\Pr(\chi^2_{33} \le 42.6779) = .879351 \tag{5.10}$$

and the coefficients $\overset{*}{\beta}_0$, $\overset{*}{\delta}_0$, $\overset{*}{\delta}_1$, $\overset{*}{\delta}_2$, given by (3.32), (3.34), (3.35), and (3.36), respectively. The computation of these coefficients from the above relations is trivial and it involves the values of α and γ_2 in (5.2) and (5.3) and t_1, t_2, and t_3 of (3.3); the latter depend on the specific alternative A_1 in the class (1.8). We specify A_1 by the following Ω–matrices.

$$\Omega_1 = \text{diag}(15, -23, 25, 30) \qquad \Omega_2 = \text{diag}(10, 20, 17, -15)$$

$$\Omega_3 = \text{diag}(9, 12, -16, 20).$$

Note that, since $m \approx 142$, an entry of Ω_i equal to 20 (for example), means that the "deviation" from the sphericity hypothesis H is 20/142 or about 14%. For these Ω's, we obtain:

$$t_1 = \frac{1}{3}\,\text{tr}(\Omega_1 + \Omega_2 + \Omega_3) = 34.6666$$

$$t_2 = \frac{1}{3}\,\text{tr}(\Omega_1^2 + \Omega_2^2 + \Omega_3^2) = 1391.7221$$

$$t_3 = \frac{1}{3}\,\text{tr}(\Omega_1^3 + \Omega_2^3 + \Omega_3^3) = 16910.6641$$

$$\overset{*}{\beta}_0 = -272.7221 \qquad \text{(from (3.32))}$$

$$\overset{*}{\delta}_0 = 35661.8555 \qquad \text{(from (3.34))}$$

$$\overset{*}{\delta}_1 = -69515.3125 \qquad \text{(from (3.35))}$$

$$\overset{*}{\delta}_2 = 33853.4570 \qquad \text{(from (3.36))}.$$

With the above values, from Theorem 1 we compute the power of the test as

$$(\text{Power at } A_1) = 1 - \Pr(-m\ell n\ \lambda^* \leq 42.6779) = .1229.$$

The power of the test when alternative A_2 is assumed is computed from Theorem 2 of Section 4. Application of this theorem requires the probabilities in (5.8) – (5.10), and the coefficients $\tilde{\beta}_0$, $\tilde{\delta}_0$, $\tilde{\delta}_1$, $\tilde{\delta}_2$ given by (4.26), (4.28), (4.29), and (4.30), respectively. As in the previous case, these coefficients depend on α, γ_2 and τ_1, τ_2, and τ_3; the latter depend on the specific alternative A_2 in the class (1.9), in particular on the Γ–matrices, which we choose as

$$\Gamma_1 = \mathrm{diag}(10, 10, -10, -10) \qquad \Gamma_2 = \mathrm{diag}(20, 15, 10, -20)$$

$$\Gamma_3 = \mathrm{diag}(10, -10, 10, 15).$$

For these Γ's we obtain:

$$\tau_1 = \frac{1}{3}\,\mathrm{tr}(\Gamma_1 + \Gamma_2 + \Gamma_3) = 16.6666$$

$$\tau_2 = \frac{1}{3}\,\mathrm{tr}(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2) = 683.3332$$

$$\tau_3 = \frac{1}{3}\,\mathrm{tr}(\Gamma_1^3 + \Gamma_2^3 + \Gamma_3^3) = 2{,}916.6665$$

$$\tilde{\beta}_0 = -153.4722 \qquad \text{(from (4.26))}$$

$$\tilde{\delta} = 10300.4961 \qquad \text{(from (4.28))}$$

$$\tilde{\delta}_1 = -21805.5469 \qquad \text{(from (4.29))}$$

$$\tilde{\delta}_2 = 11505.0508 \qquad \text{(from (4.30))}.$$

With the above values, from Theorem 2 we compute the power of the test as

$$(\text{Power at } A_2) = 1 - \Pr(-m\ell n\ \lambda^* \le 42.6779) = .0895.$$

TABLE 1. IRIS DATA

	SETOSA				VERSICOLOR				VIRGINICA			
OBS	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
1	50	33	14	2	65	28	46	15	64	28	56	22
2	46	34	14	3	62	22	45	15	67	31	56	24
3	46	36	10	2	59	32	48	18	63	28	51	15
4	51	33	17	5	61	30	46	14	69	31	51	23
5	55	35	13	2	60	27	51	16	65	30	52	20
6	48	31	16	2	56	25	39	11	65	30	55	18
7	52	34	14	2	57	28	45	13	58	27	51	19
8	49	36	14	1	63	33	47	16	68	32	59	23
9	44	32	13	2	70	32	47	14	62	34	54	23
10	50	35	16	6	64	32	45	15	77	38	67	22
11	44	30	13	2	61	28	40	13	67	33	57	25
12	47	32	16	2	55	24	38	11	76	30	66	21
13	48	30	14	3	54	30	45	15	49	25	45	17
14	51	38	16	2	58	26	40	12	67	30	52	23
15	48	34	19	2	55	26	44	12	59	30	51	18
16	50	30	16	2	50	23	33	10	63	25	50	19
17	50	32	12	2	67	31	44	14	64	32	53	23
18	43	30	11	1	56	30	45	15	79	38	64	20
19	58	40	12	2	58	27	41	10	67	33	57	21
20	51	38	19	4	60	29	45	15	77	28	67	20
21	49	30	14	2	57	26	35	10	63	27	49	18
22	51	35	14	2	57	29	42	13	72	32	60	18
23	50	34	16	4	49	24	33	10	61	30	49	18
24	46	32	16	2	56	27	42	13	61	26	56	14
25	57	44	15	4	57	30	42	12	64	28	56	21
26	50	36	14	2	66	29	46	13	62	28	48	18
27	54	34	15	4	52	27	39	14	77	30	61	23
28	52	41	15	1	60	34	45	16	63	34	56	24
29	55	42	14	2	50	20	35	10	58	27	51	19
30	49	31	15	2	55	24	37	10	72	30	58	16
31	54	39	17	4	58	27	39	12	71	30	59	21
32	50	34	15	2	62	29	43	13	64	31	55	18
33	44	29	14	2	59	30	42	15	60	30	48	18
34	47	32	13	2	60	22	40	10	63	29	56	18
35	46	31	15	2	67	31	47	15	77	26	69	23
36	51	34	15	2	63	23	44	13	60	22	50	15
37	50	35	13	3	56	30	41	13	69	32	57	23
38	49	31	15	1	63	25	49	15	74	28	61	19
39	54	37	15	2	61	28	47	12	56	28	49	20
40	54	39	13	4	64	29	43	13	73	29	63	18
41	51	35	14	3	51	25	30	11	67	25	58	18
42	48	34	16	2	57	28	41	13	65	30	58	22
43	48	30	14	1	61	29	47	14	69	31	54	21
44	45	23	13	3	56	29	36	13	72	36	61	25
45	57	38	17	3	69	31	49	15	65	32	51	20
46	51	38	15	3	55	25	40	13	64	27	53	19
47	54	34	17	2	55	23	40	13	68	30	55	21
48	51	37	15	4	66	30	44	14	57	25	50	20
49	52	35	15	2	68	28	48	14	58	28	51	24
50	53	37	15	2	67	30	50	17	63	33	60	25

(1) = sepal length (3) = petal length

(2) = sepal width (4) = petal width

ACKNOWLEDGEMENTS.

The author is grateful to a referee for discovering an error in Lemma 2 in an earlier version of the paper. He is also grateful to the editor for valuable comments which have enhanced the article's readability and for suggesting the inclusion of Section 5. The paper was presented by the author at the 17th European Meeting of Statisticians, Thessaloniki, Greece, August 1987.

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Received 10/29/85; Revised 5/24/88.