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TESTS OF HYPOTHESES ON CONCOMITANT VARIABLES IN LINEAR MODELS

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ABSTRACT

The likelihood ratio test for the equality of k univariate normal populations is extended to include the case in which the means are expressed as simple linear regression functions involving two parameters. The moments of the likelihood ratio statistic are derived, and an approximation to the null distribution is obtained.

1. INTRODUCTION

The univariate version of the problem of testing the equality of k normal populations is concerned with the hypothesis that, simultaneously, the k means μ_i are all equal, and the k variances σ_i^2 are also equal; that is,

$$H: \mu_1 = \mu_2 = \cdots = \mu_k \, ; \, \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2. \qquad (1.1)$$

Perhaps the best motivation for such a test is provided by Snedecor and Cochran (1967, p. 324): Suppose that the means μ_i are the results of various treatments applied to otherwise homogeneous units, but that the treatment effect is such that it results not only in unequal means, but also in unequal variances. In this case, testing of H is more appropriate than testing the equality of means assuming the equality of variances.

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A similar motivation leads to an extension of the above problem to the case where the means μ_i are expressed as simple linear regression functions involving two parameters. Consider the k linear regressions

$$y_{ij} = \alpha_i + \beta_i(x_{ij} - \bar{x}_{i\cdot}) + \varepsilon_{ij} \tag{1.2}$$

for $i = 1, \cdots, k$, $j = 1, \cdots, N_i$, with all the errors ε_{ij} independent and normally distributed with means 0 and variances σ_i^2, where $\bar{x}_{i\cdot} = \sum_j x_{ij}/N_i$. A hypothesis commonly tested in this case is the hypothesis that the intercepts α_i are all equal, and the slopes β_i are all equal, assuming the equality of the variances σ_i^2, $i = 1, \cdots, k$, see Graybill (1976, p. 297). This hypothesis is important; for example, in economic applications one is interested in testing the hypothesis that the same model applies, i.e., intercept and slope remain unchanged over various economic conditions. Clearly, if the conditions are such that the variances may change as well, then one needs to test the equality of not only intercepts and slopes, but the variances as well. This leads to the hypothesis

$$H_0\colon\ \alpha_i = \alpha,\ \beta_i = \beta,\ \sigma_i^2 = \sigma^2,\ i = 1, \cdots, k \tag{1.3}$$

where α, β, and σ^2 are unspecified constants.

Evidently, if the linear regression is not significant, then $\beta_i = 0$, $i = 1, \cdots, k$ and our hypothesis H_0 reduces to H. The latter was first considered in Neyman and Pearson (1931). For derivations associated directly with this univariate version, the reader is referred to Mathai and Rathie (1980), and for the multivariate analogue to Jain, Rathie and Shah (1975), and Chang, Krishnaiah and Lee (1977).

The likelihood ratio test for H_0 is derived in Section 2, and its null and non–null moments are obtained in Section 3. Section 4 gives approximations to its null distribution.

2. THE LIKELIHOOD RATIO TEST FOR H_0

Let y_{ij}, $i = 1, \cdots, k$, $j = 1, \cdots, N_i$ be distributed as in (1.2), and let $N = N_1 + N_2 + \cdots + N_k$ be the total number of observations on all k regressions. Then, the likelihood function is

$$L = L(\alpha_i, \beta_i, \sigma_i^2, i=1,\cdots,k; y_{ij}, i=1,\cdots,k, j=1,\cdots,N_i)$$

$$= \prod_{i=1}^{k} \prod_{j=1}^{N_i} \left\{(2\pi\sigma_i^2)^{-\frac{1}{2}} \exp[-\{y_{ij}-\alpha_i-\beta_i(x_{ij}-\bar{x}_{i.})\}^2/(2\sigma_i^2)]\right\}. \tag{2.1}$$

The parameter space is $\Omega = \{(\alpha_i, \beta_i, \sigma_i^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+, i=1,\cdots,k\}$. The values of α_i, β_i and σ_i^2 that maximize L under Ω are:

$$\hat{\alpha}_i = \bar{y}_{i.} = \sum_{j=1}^{N_i} y_{ij}/N_i \tag{2.2}$$

$$\hat{\beta}_i = \sum_{j=1}^{N_i} c_{ij}(y_{ij} - \bar{y}_{i.}) \tag{2.3}$$

where

$$c_{ij} = (x_{ij} - \bar{x}_{i.})/d_i \tag{2.4}$$

$$d_i = \sum_{j=1}^{N_i} (x_{ij} - \bar{x}_{i.})^2 \tag{2.5}$$

$$\hat{\sigma}_i^2 = S_i^2/N_i \tag{2.6}$$

where

$$S_i^2 = \sum_{j=1}^{N_i} [y_{ij} - \bar{y}_{i.} - \hat{\beta}_i(x_{ij} - \bar{x}_{i.})]^2. \tag{2.7}$$

Hence, the supremum of the likelihood L under Ω is

$$\sup_{\Omega} L = (2\pi)^{-N/2} \exp(-N/2) \prod_{i=1}^{k} (S_i^2 / N_i)^{-N_i/2}. \tag{2.8}$$

The parameter space specified by the null hypothesis is $\omega = \{(\alpha, \beta, \sigma^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+\}$, and the likelihood L is maximized at the following values of α, β and σ^2.

$$\hat{\alpha} = \bar{y}_{..} = \sum_{i=1}^{k} N_i \bar{y}_{i.} / N, \tag{2.9}$$

$$\hat{\beta} = \sum_{i=1}^{k} \{d_i / \sum_{j=1}^{N_i} d_j\} \hat{\beta}_i, \tag{2.10}$$

$$\hat{\sigma}^2 = S^2/N, \tag{2.11}$$

where

$$S^2 = \sum_{i=1}^{k} \sum_{j=1}^{N_i} [y_{ij} - \hat{\alpha} - \hat{\beta}(x_{ij} - \bar{x}_{i.})]^2. \tag{2.12}$$

Hence, the supremum of the likelihood L under ω is

$$\sup_{\omega} L = (2\pi)^{-N/2}(S^2/N)^{-N/2} \exp(-N/2) . \tag{2.13}$$

The likelihood ratio is now obtained from (2.8) and (2.13) as

$$\lambda = [N^{N/2}/\prod_{i=1}^{k} N_i^{N_i/2}]\prod_{i=1}^{k}(S_i^2)^{N_i/2}/(S^2)^{N/2}, \; 0 < \lambda < 1. \tag{2.14}$$

Thus, the test rejects H_0 for $\lambda \leq \lambda_{\alpha_0}$ such that, under H_0,

$$\Pr(\lambda \leq \lambda_{\alpha_0}) = \alpha_0 . \tag{2.15}$$

It will be convenient to express S^2 in terms of S_i^2, $\hat{\alpha}_i = \bar{y}_{i.}$ and $\hat{\beta}_i$. We have

$$\begin{aligned} S^2 &= \sum_i \sum_j [y_{ij} - \hat{\alpha} - \hat{\beta}(x_{ij} - \bar{x}_{i.})]^2 \\ &= \sum_i S_i^2 + \sum_i N_i(\hat{\alpha}_{i.} - \bar{y}_{..})^2 + \sum_i d_i(\hat{\beta}_i - \bar{\beta})^2, \end{aligned} \tag{2.16}$$

where

$$\bar{\beta} = \sum_i (d_i/\sum_j d_j)\, \hat{\beta}_i . \tag{2.17}$$

Thus, upon substituting S^2 from (2.16), the likelihood ratio criterion λ is expressed as a function of S_i^2, $\hat{\alpha}_i = \bar{y}_{i.}$ and $\hat{\beta}_i$, $i = 1, \cdots, k$, all mutually independent with distributions as follows:

$$S_i^2/\sigma_i^2 \sim \chi^2(n_i),\; n_i = N_i - 2 , \tag{2.18}$$

$$\hat{\alpha}_i \sim N(\alpha_i, \sigma_i^2/N_i) , \tag{2.19}$$

$$\hat{\beta}_i \sim N(\beta_i, \sigma_i^2/d_i) . \tag{2.20}$$

In summary, we have the following result.

Lemma 1. The likelihood ratio test for H_0 rejects H_0 for $\lambda \leq \lambda_{\alpha_0}$ where $\Pr(\lambda \leq \lambda_{\alpha_0}) = \alpha_0$ when H_0 is true; λ is given in (2.14) with S_i^2 and S^2 expressed as in (2.7) and (2.16) respectively.

3. MOMENTS OF THE LIKELIHOOD RATIO

In this section we derive the moments of the likelihood ratio λ. Since $0 < \lambda < 1$, it is well known that these moments uniquely determine the distribution of λ. The exact h–th moment $E(\lambda^h)$ can be obtained in the most general case by integrating out over the joint densities of (2.18) –

(2.20). This is referred to as the non–null moment (i.e. without assuming H_0 to be true). We have:

$$E(\lambda^h) = a^h \int_{S_1^2} \cdots \int_{S_k^2} \int_{\hat{\alpha}_1} \cdots \int_{\hat{\alpha}_k} \int_{\hat{\beta}_1} \cdots \int_{\hat{\beta}_k} (S^2)^{-Nh/2} \prod_{i=1}^{k} (S_i^2)^{N_i h/2}$$

$$\times \prod_{i=1}^{k} \left\{ (S_i^2)^{n_i/2-1} \exp(-S_i^2/(2\sigma_i^2)) / \{(2\sigma_i^2)^{n_i/2} \Gamma(n_i/2)\} \right\} dS_i^2$$

$$\times \prod_{i=1}^{k} [(2\pi\sigma_i^2/N_i)^{-1/2} \exp\{-N_i(\hat{\alpha}_i - \alpha_i)^2/(2\sigma_i^2)\}] \, d\hat{\alpha}_i$$

$$\times \prod_{i=1}^{k} [(2\pi\sigma_i^2/d_i)^{-1/2} \exp\{-d_i(\hat{\beta}_i - \beta_i)^2/2\sigma_i^2\}] \, d\hat{\beta}_i \tag{3.1}$$

where

$$a = N^{N/2} / \prod_{i=1}^{k} N_i^{N_i/2} . \tag{3.2}$$

To evaluate the above integral, we use the following representation:

$$(S^2)^{-Nh/2} = [\Gamma(Nh/2)]^{-1} \int_0^\infty z^{Nh/2-1} \exp(-zS^2) \, dz \text{ , } Re(Nh) > 0$$

$$= [\Gamma(Nh/2)]^{-1} \int_0^\infty z^{Nh/2-1} \exp\{-z\sum_i S_i^2 - z\sum_i N_i(\hat{\alpha}_i - \bar{y}_{..})^2$$

$$- z\sum_i d_i (\beta_i - \bar{\beta})^2\} \, dz. \tag{3.3}$$

On substituting (3.3), (3.1) becomes:

$$E(\lambda^h) = a^h [\Gamma(Nh/2)]^{-1} \int_0^\infty z^{Nh/2-1} I_1(z) I_2(z) I_3(z) dz, \tag{3.4}$$

where

$$I_1(z) = \int_{S_1^2} \cdots \int_{S_k^2} \prod_{i=1}^{k} (S_i^2)^{N_i h/2} \exp(-z \sum_i S_i^2)$$

$$\times \prod_{i=1}^{k} [(S_i^2)^{n_i/2-1} \exp(-S_i^2/(2\sigma_i^2)) / \{(2\sigma_i^2)^{n_i/2} \Gamma(n_i/2)\}] \, dS_1^2 \cdots dS_k^2 \text{ ;}$$

$$I_2(z) = \int_{\hat{\alpha}_1} \cdots \int_{\hat{\alpha}_k} \prod_{i=1}^{k} \left\{ N_i^{1/2} \exp[-N_i(\hat{\alpha}_i - \alpha_i)^2/2\sigma_i^2] (2\pi\sigma_i^2)^{-1/2} \right\}$$

$$\times \exp\{-z \sum_i N_i(\hat{\alpha}_i - \bar{y}_{..})^2\} \, d\hat{\alpha}_1 \cdots d\hat{\alpha}_k \text{ ;}$$

$$I_3(z) = \int_{\hat{\beta}_1} \cdots \int_{\hat{\beta}_k} \prod_{i=1}^{k} \left\{ d_i^{1/2} \exp[-d_i(\hat{\beta}_i - \beta_i)^2/(2\sigma_i^2)](2\pi\sigma_i^2)^{-1/2} \right\}$$

$$\times \exp\{-z \sum_i d_i(\hat{\beta}_i - \bar{\beta})^2\} \, d\hat{\beta}_1 \cdots d\hat{\beta}_k \, .$$

The integral $I_1(z)$ is easily evaluated as a product of k gamma-type integrals, i.e.,

$$I_1(z) = \prod_{i=1}^{k} \left[\int_{S_i^2>0} \frac{\left[S_i^2\right]^{N_i h/2+n_i/2-1} \; e^{-S_i^2(z+1/(2\sigma_i^2))}}{(2\sigma_i^2)^{n_i/2} \; \Gamma(n_i/2)} \, dS_i^2 \right]$$

$$= \prod_{i=1}^{k} \left\{ (z + (2\sigma_i^2)^{-1})^{-(N_i h/2 + n_i/2)} \frac{\Gamma[N_i h/2 + n_i/2]}{(2\sigma_i^2)^{n_i/2} \; \Gamma(n_i/2)} \right\} . \qquad (3.5)$$

Following Mathai and Rathie (1980), we evaluate the integral $I_2(z)$ as follows: Let,

$$w_i = (\hat{\alpha}_{i.} - \alpha_i)(N_i/(2\sigma_i^2))^{1/2} \, , \; \underset{\sim}{w} = (w_1, \cdots, w_k)' \qquad (3.6)$$

$$A = (a_{ij}), \, a_{ii} = N_i - N^{-1}N_i^2 \, , \, a_{ij} = -N_i N_j/N \, , \, i \neq j \qquad (3.7)$$

$$D = \text{diag}(\delta_1, \cdots, \delta_k) \, , \, \delta_i = (2\sigma_i^2/N_i)^{1/2} \qquad (3.8)$$

$$\underset{\sim}{\alpha} = (\alpha_1, \cdots, \alpha_k)' \, , \quad \hat{\underset{\sim}{\alpha}} = (\hat{\alpha}_1, \cdots, \hat{\alpha}_k)' . \qquad (3.9)$$

Then, we have:

$$\hat{\underset{\sim}{\alpha}} = \underset{\sim}{\alpha} + D\underset{\sim}{w}, \qquad (3.10)$$

$$\sum_i N_i(\hat{\alpha}_i - \bar{y}_{..})^2 = \hat{\underset{\sim}{\alpha}}' A \hat{\underset{\sim}{\alpha}} = \underset{\sim}{\alpha}' A \underset{\sim}{\alpha} + 2\underset{\sim}{\alpha}' A D \underset{\sim}{w} + \underset{\sim}{w}' D A D \underset{\sim}{w} \, , \qquad (3.11)$$

$$\prod_{i=1}^{k} \left\{ N_i^{1/2} \exp[-N_i(\hat{\underset{\sim}{\alpha}}_i - \alpha_i)^2/(2\sigma_i^2)](2\pi\sigma_i^2)^{-1/2} \right\} d\hat{\alpha}_1 \cdots d\hat{\alpha}_k$$

$$= e^{-\underset{\sim}{w}'\underset{\sim}{w}} \, d\underset{\sim}{w} \, /(\pi)^{k/2} . \qquad (3.12)$$

Hence,

$$I_2(z) = \pi^{k/2} \int_{\underset{\sim}{w}} e^{-z(\underset{\sim}{\alpha}' A \underset{\sim}{\alpha} + 2\underset{\sim}{\alpha}' A D \underset{\sim}{w} + \underset{\sim}{w}' D A D \underset{\sim}{w}) - \underset{\sim}{w}'\underset{\sim}{w}} \, d\underset{\sim}{w}$$

$$= |I + zDAD|^{-1/2}$$

$$\times \exp\{-z\underset{\sim}{\alpha}' A \underset{\sim}{\alpha} + z^2 \underset{\sim}{\alpha}' AD(I + zDAD)^{-1} DA\underset{\sim}{\alpha} \}. \qquad (3.13)$$

Finally, we note that the integral $I_3(z)$ is of the same form as $I_2(z)$ with d_i instead of N_i, $\hat{\beta}_i$ instead of α_i, $\bar{\beta}$ (see (2.12)) instead of $\bar{y}$.. Let,

$$u_i = (\hat{\beta}_i - \beta_i)(d_i/2\sigma_i^2)^{1/2}, \quad \underset{\sim}{u} = (u_1, \ldots, u_k)', \tag{3.14}$$

$$B = (b_{ij}),\ b_{ii} = d_i - \left[\sum_j d_j\right]^{-1} d_i^2,\ b_{ij} = -d_i d_j / \sum_j d_j,\ i \neq j, \tag{3.15}$$

$$D_1 = \operatorname{diag}(c_1, \ldots, c_k),\ c_i = (2\sigma_i^2 / \sum_j d_j)^{1/2}, \tag{3.16}$$

$$\underset{\sim}{\beta} = (\beta_1, \ldots, \beta_k)',\ \hat{\underset{\sim}{\beta}} = (\hat{\beta}_1, \ldots, \hat{\beta}_k)'. \tag{3.17}$$

Then, in complete analogy with the previous result we obtain:

$$I_3(z) = |I + zD_1BD_1|^{-1/2} \times \exp\left\{-z\underset{\sim}{\beta}'B\underset{\sim}{\beta} + z^2\underset{\sim}{\beta}'BD_1(I + zD_1BD_1)^{-1}D_1B\underset{\sim}{\beta}\right\}. \tag{3.18}$$

Thus, the non–null moment is obtained by substituting $I_1(z)$, $I_2(z)$ and $I_3(z)$ in (3.4), and it is given in the following:

Theorem 1. The h–th non–null moment of λ is given by

$$E(\lambda^h) = a^h \prod_{i=1}^k \frac{\Gamma(N_i h/2 + n_i/2)}{(2\sigma_i^2)^{n_i/2}\ \Gamma(n_i/2)}\, g$$

where $g = \int_0^\infty z^{nh/2-1} \prod_{i=1}^k \left[(z + (2\sigma_i^2)^{-1}\right]^{-(N_i h/2 + n_i/2)}$

$$\times |I + zDAD|^{-1/2} \exp\{-z\underset{\sim}{\alpha}'A\underset{\sim}{\alpha} + z^2\underset{\sim}{\alpha}'AD(I + zDAD)^{-1}DA\underset{\sim}{\alpha}\}$$

$$\times |I + zD_1BD_1|^{-1/2} \exp\{-z\underset{\sim}{\beta}'BD_1(I + zD_1BD_1)^{-1}D_1B\underset{\sim}{\beta}\}\, dz, \tag{3.19}$$

with a as in (3.2) and A, D, $\underset{\sim}{\alpha}$, B, D_1, $\underset{\sim}{\beta}$ defined in (3.7)–(3.9) and (3.15)–(3.17) respectively.

The expression for the null moments now is easily available from the above theorem. First, we note that under H_0, $\alpha_i = \alpha$, $\beta_i = \beta$, $\sigma_i^2 = \sigma^2$ for some α, β, σ^2 for $i = 1, \ldots, k$. Thus, $A\underset{\sim}{\alpha} = \underset{\sim}{0}$, $B\underset{\sim}{\beta} = \underset{\sim}{0}$, and hence, the exponential part in g is equal to 1. Also,

$$|I + zDAD| = \prod_{i=1}^{k-1} (1 + 2\sigma^2 z) = (1 + 2\sigma^2 z)^{k-1}, \tag{3.20}$$

$$|I + zD_1BD_1| = \prod_{i=1}^{k-1} (1 + 2\sigma^2 z) = (1 + 2\sigma^2 z)^{k-1}. \tag{3.21}$$

Thus, the null moment becomes

$$E(\lambda^h) = a^h[\Gamma(Nh/2)]^{-1}(2\sigma^2)^{Nh/2} \prod_{i=1}^{k} \frac{\Gamma(N_i h/2 + n_i/2)}{\Gamma(n_i/2)}$$
$$\times \int_0^\infty z^{Nh/2-1}(1 + 2\sigma^2 z)^{-(Nh/2+n/2+k-1)} dz$$

where $n = \sum_i n_i$. In the above integral we put $2\sigma^2 z = (1-v)/v$. Then, $dz = -dv/(2\sigma^2 v^2)$, and we have:

$$\int_0^\infty z^{Nh/2-1}(1 + 2\sigma^2 z)^{-(Nh/2+n/2+k-1)} dz$$
$$= (2\sigma^2)^{-Nh/2}\, \Gamma(Nh/2) \frac{\Gamma(n/2 + k - 1)}{\Gamma(Nh/2 + n/2 + k - 1)}.$$

Hence, we obtain the following result:

Theorem 2. If H_0 is assumed true, the moments of λ are given by

$$E(\lambda^h) = a^h \frac{\Gamma(n/2 + k - 1)}{\Gamma(Nh/2 + n/2 + k - 1)} \prod_{i=1}^{k} \left\{\frac{\Gamma(N_i h/2 + n_i/2)}{\Gamma(n_i/2)}\right\} \tag{3.22}$$

where $n_i = N_i - 2$, $n = \sum_{i=1}^{k} n_i$, $\quad a = N^{N/2} / \prod_{i=1}^{k} N_i^{N_i/2}$.

Replacing h/2 by h, expanding the gammas and then inverting by using the inverse Mellin transform one can get the exact density of λ^2 but the expressions will be complicated. Hence we will consider an approximation here.

4. APPROXIMATIONS TO THE NULL DISTRIBUTION

Substituting $n_i = N_i - 2$, $n = N - 2k$ in the null moment expression (3.22), we obtain:

$$E(\lambda^h) = C\, a^h \frac{\prod_{i=1}^{k} \Gamma[N_i(1+h)/2-1]}{\Gamma[N(1+h)/2-1]} \tag{4.1}$$

where C is such that $E(\lambda^0) = 1$.

Various approximations to the null distribution of λ are available by approximating the gammas in (4.1) by using the formula (see Luke (1969), Vol. I):

$$\Gamma(z+\delta) = (2\pi)^{1/2} z^{z+\delta-1/2} e^{-z-b(z,\delta)}, \tag{4.2}$$

where

$$b(z,\delta) = \sum_{r=1}^{\infty} (-1)^r B_{r+1}(\delta)/\{r(r+1)z^r\}, \tag{4.3}$$

and $B_{r+1}(\delta)$ is a Bernoulli polynomial. This formula is valid for large z and bounded δ. Applying (4.1) with $z = \frac{N_i}{2}(1+h)$ and $\delta = -1$, we obtain

$$\prod_{i=1}^{k} \Gamma[\tfrac{N_i}{2}(1+h)-1] = \prod_{i=1}^{k} \Big\{(2\pi)^{1/2} \left(\tfrac{N_i}{2}(1+h)\right)^{\frac{N_i}{2}(1+h)-\frac{3}{2}}$$

$$\times\exp[-\tfrac{N_i}{2}(1+h) - \sum_{r=1}^{\infty}(-1)^r B_{r+1}(-1)/\{r(r+1)z^r\}]\Big\}$$

$$= (2\pi)^{k/2} \prod_{i=1}^{k} [(\tfrac{N_i}{2})^{\frac{N_i}{2}(1+h)-\frac{3}{2}}](1+h)^{\frac{N}{2}(1+h)-\frac{3k}{2}}$$

$$\times\exp\left\{-\frac{N}{2}(1+h) - \sum_{i=1}^{k}\sum_{r=1}^{\infty} \frac{(-1)^r B_{r+1}(-1)}{r(r+1)[\frac{N_i}{2}(1+h)]^r}\right\}. \tag{4.4}$$

Similarly we obtain

$$\Gamma\left[\frac{N}{2}(1+h)-1\right] = (2\pi)^{1/2} [\tfrac{N}{2}]^{\frac{N}{2}(1+h)-\frac{3}{2}} (1+h)^{\frac{N}{2}(1+h)-\frac{3}{2}}$$

$$\times\exp\left\{-\frac{N}{2}(1+h) - \sum_{r=1}^{\infty} \frac{(-1)^r B_{r+1}(-1)}{r(r+1)[\frac{N}{2}(1+h)]^r}\right\}. \tag{4.5}$$

On combining (4.4) and (4.5) we obtain

$$E(\lambda^h) = C'(1+h)^{-\frac{3}{2}(k-1)}$$

$$\times\exp\left\{-\sum_{r=1}^{\infty}\frac{(-1)^r\ B_{r+1}(-1)}{r(r+1)(1+h)^r}\left[\sum_{i=1}^{k}\frac{1}{(N_i/2)^r}-\frac{1}{(N/2)^r}\right]\right\}$$

$$= C'(1+h)^{-\frac{3}{2}(k-1)}\left\{1+\frac{a_1}{N(1+h)}+\frac{a_2}{N^2(1+h)^2}+O(N^{-3})\right\}, \quad (4.6)$$

where

$$C' = \frac{(2\pi)^{\frac{k-1}{2}}\left[\prod_{i=1}^{k} N_i^{N_i/2}\right]^{-3/2}\Gamma[\frac{N}{2}-1]}{[N^{N/2}]^{-3/2}\prod_{i=1}^{k}\Gamma\ [\frac{N_i}{2}-1]},$$

$$a_1 = \frac{13}{6}\{\sum_{i=1}^{k}(1/\theta_i)-1\},$$

$$a_2 = 2\{\sum_{i=1}^{k}(1/\theta_i^2)-1\}+\frac{169}{72}\{\sum_{i=1}^{k}(1/\theta_i)-1\}$$

and $N_i = \theta_i N$. Letting now $v = -2\ln\lambda$, we have

$$E(e^{hv}) = E(e^{-2h\ln\lambda}) = E(\lambda^{-2h})$$

which is the moment generating function of v , and it is obtained from (4.6) by replacing $-2h$ for h.

$$E(e^{hv}) = C'(1-2h)^{-\frac{3}{2}(k-1)}$$

$$\times\left\{1+\frac{a_1}{N(1-2h)}+\frac{a_2}{N^2(1-2h)^2}+O(N^{-3})\right\}.$$

By inverting the above we obtain a first approximation of the density of v given by

$$g(v) = C' \frac{v^{f-1} \, e^{-(v/2)}}{2^f \, \Gamma(f)}$$

and a second approximation given by

$$g(v) = C' \left\{ \frac{v^{f-1} \, e^{-(v/2)}}{2^f \, \Gamma(f)} + \frac{a_1 \, v^f \, e^{-(v/2)}}{N \, 2^{f+1} \, \Gamma(f+1)} \right\}.$$

where $f=3(k-1)/2$.

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