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On a Multivariate Gamma

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In this paper a new form of multivariate gamma is defined whose components are positively correlated and have a three parameter gamma distribution. Explicit forms of moments, moment generating function, conditional moments, and density representations are derived. Several properties of the distribution are established. Included are also approximations, asymptotic results and Chebyshev's type inequalities. Applications of the model and estimation of parameters are discussed.

1. INTRODUCTION

The three parameter real gamma distribution with shape, scale, and location parameters α, β, and γ, respectively, has density

$$f(x; \alpha, \beta, \gamma) = \frac{(x-\gamma)^{\alpha-1} \exp(-(x-\gamma)/\beta)}{\beta^{\alpha} \Gamma(\alpha)}, \qquad x > \gamma, \alpha > 0, \beta > 0, \quad (1)$$

and zero elsewhere. This distribution, which is Type III of Pearson's system of curves, is widely applicable as a model in reliability analysis and life-distribution studies. A comprehensive account of the properties of this distribution, including characterizations and estimation of its parameters is available in Johnson and Kotz [5]. The two-parameter (shape-scale) gamma, the standard gamma ($\beta = 1, \gamma = 0$), and the exponential are clearly particular forms of (1).

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Several multivariate extensions of the standard gamma are available in the literature. Prominent among these are distributions whose marginals are standard gamma, see Johnson and Kotz [6]. In their simplest form they are constructed as follows. Let X_j's be mutually independent and have standard gamma density functions

$$f_j(x_j)=\{\Gamma(\theta_j)\}^{-1}x_j^{\theta_j-1}e^{-x_j}, \qquad x_j>0;\theta_j>0;j=0,...,k,$$

and let $Y_j=X_0+X_j, j=1,...,k$. Then, the joint distribution of $Y_1,...,Y_k$ is defined to be a multivariate gamma. The marginals are standard gamma but correlated. Krishnaiah and Rao [9] define a multivariate gamma as the joint distribution of the diagonal elements of a Wishart matrix. They call it a multivariate chi-square. In connection with rainmaking experiments Moran [14] discusses the need for multivariate models where the marginals are preferably three-parameter gammas with different parameters. In dealing with lifetimes of components of operating systems when the components are subjected to shocks Becker and Roux [1] and Steel and Roux [16] introduce bivariate gamma models. Gaver [3] builds up a multivariate gamma model from gamma marginals by a mixing procedure. Negative binomial mixtures are considered. More specifically he defines a multivariate gamma density f as one for which the Laplace transform is given by

$$L_f(s_1,...,s_k)=\left(\alpha\Big/\left[(1+\alpha)\prod_{j=1}^{m}(1+s_j)-1\right]\right)^k$$

for $k>0$, $\alpha>0$. Various forms of bivariate gammas, mostly developed from linear combinations of independent gamma variables, may be found in Eagleson [2], Ghirtis [4], Kibble [7], Miller *et al.* [11], Moran [12, 13], and Sarmanov [15].

In this article we introduce a multivariate gamma model as the joint distribution of certain linear combinations of independent gamma variables. Our model will be applicable to the following general situation. Suppose that there is a k-variate system $(X_1,...,X_k)$, where the X_j's are identically distributed except for a scaling factor. Suppose that the components are subjected to disturbances $(\varepsilon_1,...,\varepsilon_k)$ such that the new system is $(X_1+\varepsilon_1,...,X_k+\varepsilon_k)$, where the ε_j's and X_j's are mutually independently distributed gamma variables.

Our model can also describe the following situation. To start with, one has the system $(Y_1,...,Y_k)$, where the Y_j's are independent gamma variables. These could be runoffs to a dam from k different streams. The Y_j's are disturbed so that the new components are $\mathbf{Z}'=(Y_1+\delta_1X,...,Y_k+\delta_kX)$, where the δ_j's are constants and X is a new gamma random variable which is distributed independently of $Y_1,...,Y_k$, where (′) denotes the transpose. In the situation of runoffs into a dam, X could be the contribution from a new rainfall in the region and the δ_j's could be interpreted as the coefficients representing the catchment areas of the various streams. We would like to study $\mathbf{Z}$.

As another application consider a stochastic routing problem or a stochastic travelling salesman problem. Suppose that a transportation company has k routes at its disposal for going from point A to point B. Then Y_j could represent the travel time on the jth route or cost variable or any such factor for $j=1, ..., k$. Gamma models for travel time are very reasonable. A new obstruction has created a disturbance in Y_j so that Y_j is shifted to $Y_j+\delta_j X$, where δ_j is a constant and X is another gamma random variable independent of the Y_j's. Here X could be coming from a problem at a common node or a common arc of the routes. Thus the vector of interest is $\mathbf{Z}$.

In this paper we study the properties of such a model and derive: (i) moment generating function; (ii) general product moments; (iii) variances and covariances of the components; (iv) conditional moments; (v) multiple correlation; (vi) the general density; (vii) approximations and asymptotic results; (viii) estimation of parameters; and (ix) Chebyshev's type inequalities.

2. DEFINITION AND BASIC PROPERTIES

When the random variable X is distributed like (1), we will write $X \sim G(\alpha, \beta, \gamma)$. We give the following definition.

DEFINITION 1 (A multivariate gamma). Let $V_i \sim G(\alpha_i, \beta_i, \gamma_i)$, $i = 0, 1, ..., k$, where the V_i's are mutually independent and let

$$Z_i = \frac{\beta_i}{\beta_0} V_0 + V_i, \qquad i=1, ..., k.$$

The density of the vector $\mathbf{Z}' = (Z_1, ..., Z_k)$ will be called a multivariate gamma.

Several properties of the distribution will follow directly from the definition or the moment generating function of $\mathbf{Z}$. The latter is readily available from that of X in (1):

$$M_x(t) = E(e^{tX}) = \int_\gamma^\infty \frac{(x-\gamma)^{\alpha-1} \exp(-(x-\gamma)/\beta) \exp(tx)}{\Gamma(\alpha)\, \beta^\alpha}\, dx = \frac{e^{\gamma t}}{(1-\beta t)^\alpha}.$$

Hence,

$$\begin{aligned} M_z(t) &= E(e^{t_1 Z_1 + \cdots + t_k Z_k}) \\ &= E(e^{(1/\beta_0)(\beta_1 t_1 + \cdots + \beta_k t_k) V_0})\, E(e^{t_1 V_1}) \cdots E(e^{t_k V_k}) \\ &= \frac{e^{(\gamma_0/\beta_0)(\beta_1 t_1 + \cdots + \beta_k t_k)} e^{\gamma_1 t_1 + \cdots + \gamma_k t_k}}{[1 - (\beta_1 t_1 + \cdots + \beta_k t_k)]^{\alpha_0} (1 - \beta_1 t_1)^{\alpha_1} \cdots (1 - \beta_k t_k)^{\alpha_k}} \\ &= \frac{\exp[(\mathbf{g} + (\gamma_0/\beta_0)\,\mathbf{b})'\,\mathbf{t}]}{(1 - \mathbf{b}'\mathbf{t})^{\alpha_0} \prod_{i=1}^{k} (1 - \beta_i t_i)^{\alpha_i}}, \end{aligned} \tag{2}$$

where $\mathbf{b} = (\beta_1, ..., \beta_k)'$, $\mathbf{g} = (\gamma_1, ..., \gamma_k)'$, $\mathbf{t} = (t_1, ..., t_k)'$, $|\beta_i t_i| < 1$, for all i and $|\mathbf{b}'\mathbf{t}| < 1$.

The following properties are now easily obtained directly from the definition or from the moment generating function:

(i) $Z_i \sim G(\alpha_0 + \alpha_i, \beta_i, (\gamma_0/\beta_0)\,\beta_i + \gamma_i)$.

(ii) $E(Z_i) = (\alpha_0 + \alpha_i)\,\beta_i + (\gamma_0/\beta_0)\,\beta_i + \gamma_i$.

(iii) $\mathrm{Var}(Z_i) = (\alpha_0 + \alpha_i)\,\beta_i^2$.

(iv) $\mathrm{Cov}(Z_i, Z_j) = \alpha_0 \beta_i \beta_j$, $i \neq j$.

It is evident from (iv) that Z_i and Z_j for $(i \neq j)$ are positively correlated. Reproductive properties are stated in the following.

THEOREM 2.1. *The class of multivariate gamma is closed under*: (a) *transformations of the form* $\mathbf{W} = \mathbf{Z} + \mathbf{d}$, $\mathbf{d}' = (d_1, ..., d_k)$; (b) *convolutions* $\mathbf{Z}_1 + \mathbf{Z}_2$, *where* $\mathbf{Z}_1, \mathbf{Z}_2$ *are independent*, $\mathbf{Z}_1$ *is multivariate gamma with parameters* $\alpha_i, \beta_i, \gamma_i, i = 0, 1, ..., k$, *and* $\mathbf{Z}_2$ *is multivariate gamma with parameters* $\alpha_i', \beta_i, \gamma_i', i = 0, 1, ..., k$.

Proof. (a) The moment generating function of $\mathbf{W}$ is

$$M_{\mathbf{w}}(\mathbf{t}) = \exp\left(\sum_{i=1}^{k} d_i t_i\right) M_{\mathbf{z}}(\mathbf{t})$$

which is of the form (2) with $\gamma_i + d_i$ in place of γ_i for $i = 1, ..., k$.

(b) The moment generating function of $\mathbf{Z}_1 + \mathbf{Z}_2$ is

$$M_{\mathbf{z}_1 + \mathbf{z}_2}(\mathbf{t}) = M_{\mathbf{z}_1}(\mathbf{t})\, M_{\mathbf{z}_2}(\mathbf{t})$$

and hence it is of the form (2) with the same β_i parameters, $\alpha_i + \alpha_i'$ in place of α_i and $\gamma_i + \gamma_i'$ in place of γ_i for $i = 0, 1, ..., k$.

3. MOMENTS AND CUMULANTS

Moments of the form $E(Z_i^m)$ and product moments $E(Z_i^m Z_j^r)$ will be evaluated directly from the definition. Their evaluation requires the moments $E(V_i^m)$ which are available from the moment generating function of V_i. We have

$$M_{v_i}(t) = e^{\gamma_i t}(1-\beta_i t)^{-\alpha_i}$$

$$\frac{d^m M_{v_i}(t)}{dt^m} = \sum_{k_1=0}^{m} \binom{m}{k_1} \left(\left\{ \frac{d^{k_1}}{dt^{k_1}} (1-\beta_i t)^{-\alpha_i} \right\} \left\{ \frac{d^{m-k_1}}{dt^{m-k_1}} e^{(\gamma_i t)} \right\} \right).$$

Hence, putting $t=0$ we obtain

$$M_i^{(m)} = E(V_i^m) = \sum_{k_1=0}^{m} \binom{m}{k_1} \alpha_i(\alpha_i+1)\cdots(\alpha_i+k_1-1)\,\beta_i^{k_1}\gamma_i^{m-k_1}$$

$$= \sum_{k_1=0}^{m} \binom{m}{k_1} (\alpha_i)_{k_1} \beta_i^{k_1} \gamma_i^{m-k_1}, \tag{3}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$. Using the above, we now have

$$E(Z_i^m) = E\left(\frac{\beta_i}{\beta_0} V_0 + V_i\right)^m = E\left\{ \sum_{r=0}^{m} \binom{m}{r} \left(\frac{\beta_i}{\beta_0}\right)^r V_0^r V_i^{m-r} \right\}$$

$$= \sum_{r=0}^{m} \binom{m}{r} \left(\frac{\beta_i}{\beta_0}\right)^r E(V_0)^r\, E(V_i)^{m-r}$$

$$= \sum_{r=0}^{m} \binom{m}{r} \left(\frac{\beta_i}{\beta_0}\right)^r \sum_{k_0=0}^{r} \binom{r}{k_0} (\alpha_0)_{k_0} \beta_0^{k_0} \gamma_0^{r-k_0}$$

$$\times \sum_{k_i=0}^{m-r} \binom{m-r}{k_i} (\alpha_i)_{k_i} \beta_i^{k_i} \gamma_i^{m-r-k_i}. \tag{4}$$

By the same procedure as above one also can derive the product moments directly. For example,

$$E(Z_i^m Z_j^n) = E\left(\left(\frac{\beta_i}{\beta_0} V_0 + V_i\right)^m \left(\frac{\beta_j}{\beta_0} V_0 + V_j\right)^n \right)$$

$$= E\left(\sum_{r=0}^{m} \binom{m}{r} \left(\frac{\beta_i}{\beta_0}\right)^r V_0^r V_i^{m-r} \sum_{s=0}^{n} \binom{n}{s} \left(\frac{\beta_j}{\beta_0}\right)^s V_0^s V_j^{n-s} \right)$$

$$= \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \left(\frac{\beta_i}{\beta_0}\right)^r \left(\frac{\beta_j}{\beta_0}\right)^s (EV_0^{r+s})(EV_i^{m-r})(EV_j^{n-s})$$

$$= \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \left(\frac{\beta_i}{\beta_0}\right)^r \left(\frac{\beta_j}{\beta_0}\right)^s M_0^{(r+s)} M_i^{(m-r)} M_j^{(n-s)}, \tag{5}$$

where $M_i^{(m)}$ for $i=1, ..., k$ are available from (3). To compute the cumulants of Z we need the logarithm of the moment generating function. From (2) we have

$$\ln M_z(t) = -\alpha_0 \ln\left(1-\sum_{i=1}^{k} \beta_i t_i\right) - \sum_{i=1}^{k} \alpha_i \ln(1-\beta_i t_i) + \sum_{i=1}^{k} \left(\frac{\gamma_0}{\beta_0}\beta_i + \gamma_i\right) t_i.$$

From this it easily follows that for $m_i \geqslant 2$ the cumulants of Z are

$$\begin{aligned} K_{m_i} &= \frac{\partial^{m_i}}{\partial t_i^{m_i}} (\ln M_{\mathbf{z}}(\mathbf{t}))\Big|_{\mathbf{t}=0} \\ &= \left\{\frac{\alpha_0 (m_i-1)!\, \beta_i^{m_i}}{(1-\beta_1 t_1 - \cdots - \beta_k t_k)^{m_i}} + \frac{\alpha_i (m_i-1)!\, \beta_i^{m_i}}{(1-\beta_i t_i)^{m_i}}\right\}\Big|_{t=0} \\ &= (m_i-1)!\, \beta_i^{m_i}(\alpha_0+\alpha_i). \end{aligned} \tag{6}$$

Also, the joint cumulants are obtained as

$$K_{m_i m_j} = \frac{\partial^{m_i+m_j}}{\partial t_j^{m_j} \partial t_i^{m_i}} (\ln M_{\mathbf{z}}(\mathbf{t}))\Big|_{\mathbf{t}=0} = \alpha_0(m_i+m_j-1)!\, \beta_i^{m_i} \beta_j^{m_j}. \tag{7}$$

The last equation gives, for example,

$$K_{20} = (\alpha_0+\alpha_i)\, \beta_i^2 = \operatorname{Var}(Z_i)$$

$$K_{11} = \alpha_0 \beta_i \beta_j = \operatorname{Cov}(Z_i, Z_j), \qquad i \neq j.$$

From these one can write the covariance matrix of $\mathbf{Z}' = (Z_1, ..., Z_k)$, denoted by Σ, as

$$\operatorname{Cov}(\mathbf{Z}) = \Sigma = (\sigma_{ij}) = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$$\sigma_{ii} = (\alpha_0+\alpha_i)\, \beta_i^2, \qquad \sigma_{ij} = \alpha_0 \beta_i \beta_j, \qquad i \neq j, \Sigma_{11} = \sigma_{11} = (\alpha_0+\alpha_1)\, \beta_1^2.$$

If the multiple correlation of Z_1 on $Z_2, ..., Z_k$ is denoted by $R_{1(2\cdots k)}$, then

$$1 - R^2_{1(2\cdots k)} = \frac{|\Sigma|}{\sigma_{11}\, |\Sigma_{22}|}.$$

This can be shown to have a very simple form in our model. Note that $\Sigma = DAD$, where $D = \operatorname{diag}(\beta_1, ..., \beta_k)$, $A = (a_{ij})$, $a_{ii} = \alpha_0 + \alpha_i$, $a_{ij} = \alpha_0$, $i \neq j$. Thus $|\Sigma| = \beta_1^2 \cdots \beta_k^2\, |A|$ and

$$\begin{aligned} |A| &= (\alpha_2 \cdots \alpha_k)[(\alpha_0+\alpha_1) + \alpha_0\alpha_1(1/\alpha_2 + \cdots + 1/\alpha_k)] \\ &= \sum_{(1)} (\alpha_0 \alpha_1 \cdots \alpha_k) \\ &= (\alpha_1\alpha_2\cdots\alpha_k) + (\alpha_0\alpha_2\cdots\alpha_k) + \cdots + (\alpha_0\alpha_1\cdots\alpha_{k-1}), \end{aligned}$$

where $\sum_{(1)}(\alpha_0\alpha_1\cdots\alpha_k)$ denotes the sum of all products of $\alpha_0\alpha_1\cdots\alpha_k$ deleting one of the α_j's each time. Thus

$$|\Sigma| = \beta_1^2\cdots\beta_k^2\sum_{(1)}(\alpha_0\alpha_1\cdots\alpha_k)$$

and then

$$|\Sigma_{22}| = \beta_2^2\cdots\beta_k^2\sum_{(1)}(\alpha_0\alpha_2\cdots\alpha_k).$$

Hence

$$1-R^2_{1(2\cdots k)} = (\alpha_0+\alpha_1)^{-1}\sum_{(1)}(\alpha_0\alpha_1\cdots\alpha_k)\Big/\sum_{(1)}(\alpha_0\alpha_2\cdots\alpha_k). \tag{8}$$

4. CONDITIONAL DENSITIES AND CONDITIONAL EXPECTATIONS

In this section we derive conditional expectations of the type $E(Z_1\,|\,Z_2)$ and conditional variances $\operatorname{Var}(Z_1\,|\,Z_2)$. These will be obtained from Theorem 4.1 below that gives the rth conditional moment $E(Z_1^r\,|\,Z_2)$. We will also indicate that the same techniques can lead to the evaluation of moments of the type $E(Z_1^{k_1}Z_2^{k_2}\,|\,Z_3)$ and we will illustrate the evaluation for $k_1=k_2=1$.

THEOREM 4.1. *Let* $\omega=(\beta_0/\beta_2)(z_2-\gamma_2)-\gamma_0$ *and* $\delta=\gamma_0+(\beta_0/\beta_1)\gamma_1$. *Then, the* r*th conditional moment* $E(Z_1^r\,|\,Z_2)$ *is given by*

$$E(Z_1^r\,|\,Z_2)=\frac{\beta_1^r}{\Gamma(\alpha_1)}\sum_{r_1=0}^{r}\binom{r}{r_1}\frac{\Gamma(r-r_1+\alpha_1)}{\beta_0^{r_1}}$$
$$\times\sum_{r_2=0}^{r_1}\binom{r_1}{r_2}\frac{(\alpha_0)_{r_2}}{(\alpha_0+\alpha_2)_{r_2}}\,\delta^{r_1-r_2}\omega^{r_2}.$$

Proof. Consider the joint density of V_0, V_1, and V_2,

$$f(v_0,v_1,v_2)=\prod_{i=0}^{2}\left\{\frac{(v_i-\gamma_i)^{\alpha_i-1}\exp(-(v_i-\gamma_i)/\beta_i)}{\Gamma(\alpha_i)\,\beta_i^{\alpha_i}}\right\}.$$

The joint density of $Z_1=(\beta_1/\beta_0)\,V_0+V_1$, $Z_2=(\beta_2/\beta_0)\,V_0+V_2$ is

$$g_{12}(z_1,z_2)=c\int_{v_0}(v_0-\gamma_0)^{\alpha_0-1}\,e^{-(v_0-\gamma_0)/\beta_0}\left(z_1-\frac{\beta_1}{\beta_0}v_0-\gamma_1\right)^{\alpha_1-1}$$
$$\times e^{-(z_1-(\beta_1/\beta_0)v_0-\gamma_1)/\beta_1}\left(z_2-\frac{\beta_2}{\beta_0}v_0-\gamma_2\right)^{\alpha_2-1}$$
$$\times e^{-(z_2-(\beta_2/\beta_0)v_0-\gamma_2)/\beta_2}\,dv_0,$$

where

$$c=\left\{\prod_{i=0}^{2}\Gamma(\alpha_i)\,\beta_i^{\alpha_i}\right\}^{-1}.$$

Since $Z_2\sim G(\alpha_0+\alpha_2,\beta_2,\gamma_2+\gamma_0(\beta_2/\beta_0))$, the conditional density of $(Z_1\,|\,Z_2)$ is

$$\begin{aligned}g_{1|2}(z_1\,|\,z_2)&=\frac{c_1}{(z_2-\gamma_2-\gamma_0(\beta_2/\beta_0))^{\alpha_0+\alpha_2-1}\,e^{-(z_2-\gamma_2-\gamma_0(\beta_2/\beta_0))/\beta_2}}\\&\quad\times\int_{v_0}(v_0-\gamma_0)^{\alpha_0-1}\\&\quad\times e^{-(v_0-\gamma_0)/\beta_0}\prod_{j=1}^{2}\left\{\left(z_j-\frac{\beta_j}{\beta_0}v_0-\gamma_j\right)^{\alpha_j-1}e^{-(z_j-(\beta_j/\beta_0)v_0-\gamma_j)/\beta_j}\right\}dv_0,\end{aligned}\tag{9}$$

where

$$c_1=\frac{\beta_2^{\alpha_0}\Gamma(\alpha_0+\alpha_2)}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\Gamma(\alpha_0)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)}.$$

Now, to evaluate $E(Z_1^r\,|\,Z_2)$, we first express Z_1^r as

$$\begin{aligned}Z_1^r&=\left[\left(Z_1-\frac{\beta_1}{\beta_0}V_0-\gamma_1\right)+\left(\frac{\beta_1}{\beta_0}V_0+\gamma_1\right)\right]^r\\&=\sum_{r_1=0}^{r}\binom{r}{r_1}\left(Z_1-\frac{\beta_1}{\beta_0}V_0-\gamma_1\right)^{r-r_1}\left(\frac{\beta_1}{\beta_0}V_0+\gamma_1\right)^{r_1}.\end{aligned}$$

Then, interchanging integrals in (9) we obtain

$$\begin{aligned}E(Z_1^r\,|\,Z_2)&=\frac{c_1\sum_{r_1=0}^{r}\binom{r}{r_1}\int_{v_0}(v_0-\gamma_0)^{\alpha_0-1}e^{-(v_0-\gamma_0)/\beta_0}((\beta_1/\beta_0)\,v_0+\gamma_1)^{r_1}}{(z_2-\gamma_2-\gamma_0(\beta_2/\beta_0))^{\alpha_0+\alpha_2-1}\,e^{-(z_2-\gamma_2-\gamma_0(\beta_2/\beta_0))/\beta_2}}\\&\quad\times\left(z_2-\frac{\beta_2}{\beta_0}v_0-\gamma_2\right)^{\alpha_2-1}e^{-(z_2-(\beta_2/\beta_0)v_0-\gamma_2)/\beta_2}\\&\quad\times\int_{z_1}\left(z_1-\frac{\beta_1}{\beta_0}v_0-\gamma_1\right)^{r-r_1+\alpha_1-1}e^{-(z_1-(\beta_1/\beta_0)v_0-\gamma_1)/\beta_1}\,dz_1\,dv_0\\&=\frac{c_1\sum_{r_1=0}^{r}\binom{r}{r_1}\Gamma(r-r_1+\alpha_1)\,\beta_1^{r-r_1+\alpha_1}\times A}{(z_2-\gamma_2-\gamma_0(\beta_2/\beta_0))^{\alpha_0+\alpha_2-1}\,e^{-(z_2-\gamma_2-\gamma_0(\beta_2/\beta_0))/\beta_2}},\end{aligned}\tag{10}$$

where

$$A=\int_{v_0}(v_0-\gamma_0)^{\alpha_0-1}\,e^{-(v_0-\gamma_0)/\beta_0}\left(\frac{\beta_1}{\beta_0}v_0+\gamma_1\right)^{r_1}$$
$$\times\left(z_2-\frac{\beta_2}{\beta_0}v_0-\gamma_2\right)^{\alpha_2-1}e^{-(z_2-(\beta_2/\beta_0)v_0-\gamma_2)/\beta_2}\,dv_0.$$

The last integral is easily evaluated. Let $y=v_0-\gamma_0$, Then,

$$z_2-\frac{\beta_2}{\beta_0}v_0-\gamma_2=\frac{\beta_2}{\beta_0}(\omega-y),$$

where $0<y<\omega$, and hence the integral becomes

$$A=e^{-\omega/\beta_0}\left(\frac{\beta_1}{\beta_0}\right)^{r_1}\left(\frac{\beta_2}{\beta_0}\omega\right)^{\alpha_2-1}\int_0^{\omega}y^{\alpha_0-1}(y+\delta)^{r_1}\left(1-\frac{y}{\omega}\right)^{\alpha_2-1}dy,$$

where $\delta=\gamma_0+(\beta_0/\beta_1)\gamma_1$. Finally, the transformation $y=\omega t, 0<t<1$, gives

$$A=e^{-\omega/\beta_0}\left(\frac{\beta_1}{\beta_0}\right)^{r_1}\left(\frac{\beta_2}{\beta_0}\right)^{\alpha_2-1}\omega^{\alpha_0+\alpha_2-1}\int_0^1 t^{\alpha_0-1}(1-t)^{\alpha_2-1}(\omega t+\delta)^{r_1}\,dt$$
$$=e^{-\omega/\beta_0}\left(\frac{\beta_1}{\beta_0}\right)^{r_1}\left(\frac{\beta_2}{\beta_0}\right)^{\alpha_2-1}\omega^{\alpha_0+\alpha_2-1}\Gamma(\alpha_2)$$
$$\times\sum_{r_2=0}^{r_1}\binom{r_1}{r_2}\frac{\Gamma(\alpha_0+r_2)}{\Gamma(\alpha_0+\alpha_2+r_2)}\delta^{r_1-r_2}\omega^{r_2}.$$

Upon substituting in (10) and simplifying, we obtain the result as stated in the theorem.

The conditional expectation $E(Z_1\,|\,Z_2=z_2)$ which may be called the regression of Z_1 on Z_2 is immediately obtained from the theorem for $r=1$.

COROLLARY 1. *The conditional expectation* $E(Z_1\,|\,Z_2=z_2)$ *is the following linear function in* z_2

$$E(Z_1\,|\,Z_2=z_2)=B_0+B_1(z_2-E(Z_2)),$$

where

$$B_0=E(Z_1)=(\alpha_0+\alpha_1)\beta_1+\gamma_1+\frac{\gamma_0}{\beta_0}\beta_1$$

$$B_1=\frac{\operatorname{Cov}(Z_1,Z_2)}{\operatorname{Var}(Z_2)}=\frac{\alpha_0\beta_1}{\beta_2(\alpha_0+\alpha_2)}.$$

Conditional variances of the type $\mathrm{Var}(Z_1 \mid Z_2 = z_2)$ are available from the general conditional moment $E(Z_1^r \mid Z_2 = z_2)$ given by Theorem 4.1. Hence, we will consider moments of the type $E(Z_1^{k_1} Z_2^{k_2} \mid Z_3 = z_3)$. The same technique will work for any k_1, k_2; hence, we will illustrate the method for $k_1 = k_2 = 1$. The marginal density of Z_3 is given by

$$g(z_3) = \frac{1}{\beta_3^{\alpha_0+\alpha_3} \Gamma(\alpha_0+\alpha_3)} \left(z_3 - \frac{\beta_3}{\beta_0} \gamma_0 - \gamma_3 \right)^{\alpha_0+\alpha_3-1} e^{-(z_3-(\beta_3/\beta_0)\gamma_0-\gamma_3)/\beta_3}.$$

Consider the joint density of V_0, Z_1, Z_2, Z_3,

$$f(v_0, z_1, z_2, z_3) = \frac{(v_0-\gamma_0)^{\alpha_0-1} e^{-(v_0-\gamma_0)/\beta_0}}{\prod_{j=0}^{3} \beta_j^{\alpha_j} \Gamma(\alpha_j)} \times \prod_{j=1}^{3} \left\{ \left(z_j - \frac{\beta_j}{\beta_0} v_0 - \gamma_j \right)^{\alpha_j-1} e^{-(z_j-(\beta_j/\beta_0)v_0-\gamma_j)/\beta_j} \right\}.$$

The joint conditional density of Z_1, Z_2 given $Z_3 = z_3$ is

$$g_3(z_1, z_2 \mid z_3) = \int_{v_0} \frac{f(v_0, z_1, z_2, z_3)\, dv_0}{g(z_3)}.$$

Hence,

$$E(Z_1 Z_2 \mid Z_3 = z_3) = \frac{1}{g(z_3)} \int_{v_0} \int_{z_1} \int_{z_2} z_1 z_2 f(v_0, z_1, z_2, z_3)\, dz_1\, dz_2\, dv_0. \quad (11)$$

Let

$$z_j = \left(z_j - \frac{\beta_j}{\beta_0} v_0 - \gamma_j \right) + \left(\frac{\beta_j}{\beta_0} v_0 + \gamma_j \right), \qquad j = 1, 2.$$

Then,

$$z_1 z_2 = \left(z_1 - \frac{\beta_1}{\beta_0} v_0 - \gamma_1 \right)\left(z_2 - \frac{\beta_2}{\beta_0} v_0 - \gamma_2 \right) + \left(z_1 - \frac{\beta_1}{\beta_0} v_0 - \gamma_1 \right)\left(\frac{\beta_2}{\beta_0} v_0 + \gamma_2 \right)$$
$$+ \left(z_2 - \frac{\beta_2}{\beta_0} v_0 - \gamma_2 \right)\left(\frac{\beta_1}{\beta_0} v_0 + \gamma_1 \right) + \left(\frac{\beta_1}{\beta_0} v_0 + \gamma_1 \right)\left(\frac{\beta_2}{\beta_0} v_0 + \gamma_2 \right).$$

Using the above, the integral (11) is evaluated upon evaluating the four integrals corresponding to the four terms in the expression for $z_1 z_2$. Since these integrals are of the same type, we illustrate the evaluation of one of them corresponding to the first term. All others are evaluated in a similar way:

$$
\begin{aligned}
A_1 &= \frac{1}{g(z_3)} \int_{v_0} \int_{z_1} \int_{z_2} \left(z_1 - \frac{\beta_1}{\beta_0} v_0 - \gamma_1 \right) \left(z_2 - \frac{\beta_2}{\beta_0} v_0 - \gamma_2 \right) \\
&\quad \times \frac{(v_0 - \gamma_0)^{\alpha_0 - 1} e^{-(v_0 - \gamma_0)/\beta_0}}{\prod_{j=0}^{3} \beta_j^{\alpha_j} \Gamma(\alpha_j)} \\
&\quad \times \prod_{j=1}^{3} \left\{ \left(z_j - \frac{\beta_j}{\beta_0} v_0 - \gamma_j \right)^{\alpha_j - 1} e^{-(z_j - (\beta_j/\beta_0) v_0 - \gamma_j)/\beta_j} \right\} dz_1 \, dz_2 \, dv_0 \\
&= \frac{\prod_{j=1}^{2} \{ \beta_j^{\alpha_j + 1} \Gamma(\alpha_j + 1) \}}{\prod_{j=0}^{3} \beta_j^{\alpha_j} \Gamma(\alpha_j) \, g(z_3)} \int_{v_0} (v_0 - \gamma_0)^{\alpha_0 - 1} \\
&\quad \times e^{-(v_0 - \gamma_0)/\beta_0} \left(z_3 - \frac{\beta_3}{\beta_0} v_0 - \gamma_3 \right)^{\alpha_3 - 1} e^{-(z_3 - (\beta_3/\beta_0) v_0 - \gamma_3)/\beta_3} \, dv_0 \\
&= \frac{\alpha_1 \alpha_2 \beta_1 \beta_2 (\beta_3/\beta_0)^{\alpha_3 - 1} \omega_3^{\alpha_0 + \alpha_3 - 1} e^{-\omega_3/\beta_0}}{\beta_0^{\alpha_0} \beta_3^{\alpha_3} \Gamma(\alpha_0) \, \Gamma(\alpha_3) \, g(z_3)} \int_0^1 t^{\alpha_0 - 1} (1 - t)^{\alpha_3 - 1} \, dt \\
&\qquad \left(\text{where } \omega_3 = \frac{\beta_0}{\beta_3} (z_3 - \gamma_3) - \gamma_0 \right) \\
&= \frac{\alpha_1 \alpha_2 \beta_1 \beta_2 (\beta_3/\beta_0)^{\alpha_3 - 1} \omega_3^{\alpha_0 + \alpha_3 - 1} e^{-\omega_3/\beta_0} \Gamma(\alpha_0) \, \Gamma(\alpha_3)}{\beta_0^{\alpha_0} \beta_3^{\alpha_3} \Gamma(\alpha_0) \, \Gamma(\alpha_3) \, g(z_3) \, \Gamma(\alpha_0 + \alpha_3)} = \alpha_1 \alpha_2 \beta_1 \beta_2 .
\end{aligned}
$$

Similarly, the other three integrals can be evaluated to compute $E(Z_1 Z_2 | Z_3)$. It should be noted that the same procedure will work for $E(Z_1^{k_1} Z_2^{k_2} | Z_3 = z_3)$, once $Z_1^{k_1}$ and $Z_2^{k_2}$ are expressed in terms of the powers of $z_j - (\beta_j/\beta_0) v_0 - \gamma_j, j = 1, 2$. Since the expressions will be complicated these will not be given here.

5. THE DENSITY OF THE MULTIVARIATE GAMMA

In this section we consider representations for the density of the multivariate gamma. Consider the transformations

$$
Z_i = \frac{\beta_i}{\beta_0} V_0 + V_i, \qquad i = 1, ..., k, \qquad Z_{k+1} = V_0 .
$$

From the joint density of the $k+1$ mutually independent random variables $V_0, V_1, ..., V_k$, we can obtain the joint density of $Z_1, ..., Z_k, Z_{k+1}$. Note that the transformation above has Jacobian $= 1$ and that

$$
Z_i - \frac{\beta_i}{\beta_0} Z_{k+1} = V_i, \qquad i = 1, ..., k.
$$

Thus, the joint density of $Z_1, Z_2, ..., Z_{k+1}$ is

$$\begin{aligned} &g_{k+1}(z_1, ..., z_k, z_{k+1}) \\ &\quad = C(z_{k+1}-\gamma_0)^{\alpha_0-1} \exp\left(-\frac{z_{k+1}-\gamma_0}{\beta_0}\right) \\ &\qquad \times \prod_{j=1}^{k} \left(z_j-\frac{\beta_j}{\beta_0} z_{k+1}-\gamma_j\right)^{\alpha_j-1} \exp\left\{-\left(z_j-\frac{\beta_j}{\beta_0} z_{k+1}-\gamma_j\right)\Big/\beta_j\right\}, \end{aligned} \tag{12}$$

where

$$C=\left\{\beta_0^{\alpha_0}\Gamma(\alpha_0) \prod_{j=1}^{k} \beta_j^{\alpha_j}\Gamma(\alpha_j)\right\}^{-1}.$$

In (12), put $u=z_{k+1}-\gamma_0$, $u_j=(\beta_0/\beta_j)(z_j-\gamma_j)-\gamma_0, j=1, ..., k$. Then

$$\begin{aligned} g_{k+1}(u_1, ..., u_k, u) = C \prod_{j=1}^{k} \left(\frac{\beta_j}{\beta_0}\right)^{\alpha_j-1} u^{\alpha_0-1} \exp\left(-\frac{u}{\beta_0}\right) \\ \times \prod_{j=1}^{k} \left\{(u_j-u)^{\alpha_j-1} \exp\left(-\frac{u_j-u}{\beta_0}\right)\right\}, \end{aligned}$$

where $0<u<\min\{u_1, ..., u_k\}$. The joint density of $u_1, ..., u_k$ is now available by integrating out u, i.e.,

$$\begin{aligned} g_k(u_1, ..., u_k) = C_1 \int_0^{\min\{u_1, ..., u_k\}} u^{\alpha_0-1} \exp\left(-\frac{u}{\beta_0}\right) \\ \times \prod_{j=1}^{k} \left\{(u_j-u)^{\alpha_j-1} \exp\left(-\frac{u_j-u}{\beta_0}\right)\right\} du, \end{aligned} \tag{13}$$

where

$$C_1 = C \prod_{j=1}^{k} \left(\frac{\beta_j}{\beta_0}\right)^{\alpha_j-1}.$$

It follows from the above that the density is of different form for each of the $k!$ orderings of $u_1, ..., u_k$. For example, for $u_1<u_2< \cdots <u_k$,

$$\begin{aligned} &g_k(u_1, ..., u_k) \\ &\quad = C_1 \int_0^{u_1} u^{\alpha_0-1} \exp\left(-\frac{u}{\beta_0}\right) (u_1-u)^{\alpha_1-1} \cdots (u_k-u)^{\alpha_k-k} \\ &\qquad \times \exp(-[(u_1-u)+ \cdots +(u_k-u)]/\beta_0)\, du \end{aligned}$$

$$= C_1 \left\{ \prod_{j=1}^{k} u_j^{\alpha_j - 1} \right\} \int_0^{u_1} u^{\alpha_0 - 1} \exp(-u/\beta_0) \left(1 - \frac{u}{u_1}\right)^{\alpha_1 - 1} \cdots \left(1 - \frac{u}{u_k}\right)^{\alpha_k - 1}$$

$$\times \exp\left\{ -\frac{1}{\beta_0} \left[u_1 \left(1 - \frac{u}{u_1}\right) + \cdots + u_k \left(1 - \frac{u}{u_k}\right) \right] \right\} du$$

$$= C_1 \left\{ \prod_{j=1}^{k} u_j^{\alpha_j - 1} \right\} u_1^{\alpha_0} \int_0^1 y^{\alpha_0 - 1} \exp\left(-\frac{u_1}{\beta_0} y \right) (1-y)^{\alpha_1 - 1}$$

$$\times \left(1 - \frac{u_1}{u_2} y\right)^{\alpha_2 - 1} \cdots \left(1 - \frac{u_1}{u_k} y\right)^{\alpha_k - 1}$$

$$\times \exp\left\{ -\frac{1}{\beta_0} \left[u_1(1-y) + u_2 \left(1 - \frac{u_1}{u_2} y\right) + \cdots + u_k \left(1 - \frac{u_1}{u_k} y\right) \right] \right\} dy.$$

Expanding the exponentials in series forms we obtain

$$g_k(u_1, ..., u_k)$$

$$= C_1 \left\{ \prod_{j=1}^{k} u_j^{\alpha_j - 1} \right\}$$

$$\times u_1^{\alpha_0} \sum_{r_0=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} \frac{(-u_1/\beta_0)^{r_0}}{r_0!} \frac{(-u_1/\beta_0)^{r_1}}{r_1!} \cdots \frac{(-u_k/\beta_0)^{r_k}}{r_k!}$$

$$\times \int_0^1 y^{\alpha_0 + r_0 - 1} (1-y)^{\alpha_1 + r_1 - 1}$$

$$\times \left(1 - \frac{u_1}{u_2} y\right)^{\alpha_2 + r_2 - 1} \cdots \left(1 - \frac{u_1}{u_k} y\right)^{\alpha_k + r_k - 1} dy.$$

Since $u < u_1/u_j < 1, j = 1, ..., k$,

$$g_k(u_1, ..., u_k)$$

$$= \frac{\prod_{j=1}^{k} (\beta_j/\beta_0)^{\alpha_j - 1}}{\prod_{j=0}^{k} [\beta_j^{\alpha_j} \Gamma(\alpha_j)]} \left\{ \prod_{j=1}^{k} u_j^{\alpha_j - 1} \right\} u_1^{\alpha_0}$$

$$\times \sum_{r_0=1}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} \frac{(-u_1/\beta_0)^{r_0}}{r_0!} \frac{(-u_1/\beta_0)^{r_1}}{r_1!} \cdots$$

$$\times \frac{(-u_k/\beta_0)^{r_k}}{r_k!} \frac{\Gamma(\alpha_0 + r_0)\, \Gamma(\alpha_1 + r_1)}{\Gamma(\alpha_0 + r_0 + \alpha_1 + r_1)}$$

$$\times F_D \left(\alpha_0 + r_0; \alpha_2 + r_2, ..., \alpha_k + r_k; \alpha_0 + r_0 + \alpha_1 + r_1; \frac{u_1}{u_2}, ..., \frac{u_1}{u_k} \right),$$

where F_D is the Lauricella function, see Mathai and Saxena [10].

Now, let $(i_1, i_2, ..., i_k)$ be a permutation of the integers $(1, 2, ..., k)$ such that $u_{i_1} < u_{i_2} < \cdots < u_{i_k}$. Then, the density has the form:

$$g_k(u_{i_1}, ..., u_{i_k}) = \frac{\prod_{j=1}^{k} (\beta_j/\beta_0)^{\alpha_j - 1}}{\prod_{j=0}^{k} [\beta_j^{\alpha_j} \Gamma(\alpha_j)]} \left\{ \prod_{j=1}^{k} u_j^{\alpha_j - 1} \right\}$$

$$\times u_{i_1}^{\alpha_0} \sum_{r_0=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} \frac{(-u_{i_1}/\beta_0)^{r_0}}{r_0!} \frac{(-u_{i_1}/\beta_0)^{r_1}}{r_1!} \cdots$$

$$\times \frac{(-u_{i_k}/\beta_0)^{r_k}}{r_k!} \frac{\Gamma(\alpha_0 + r_0)\, \Gamma(\alpha_1 + r_1)}{\Gamma(\alpha_0 + r_0 + \alpha_1 + r_1)}$$

$$\times F_D\left(\alpha_0 + r_0; \alpha_{i_2} + r_2, ..., \alpha_{i_k} + r_k; \alpha_0 + r_0 + \alpha_{i_1} + r_1; \frac{u_{i_1}}{u_{i_2}}, ..., \frac{u_{i_1}}{u_{i_k}}\right).$$

For each permutation $(i_1, i_2, ..., i_k)$ of the integers $(1, 2, ..., k)$ such that $u_{i_1} < u_{i_2} < \cdots < u_{i_k}$, we have a part of the density in the above form. The Lauricella function F_D has a convergent series representation for $|u_{i_1}/u_{i_j}| < 1, j = 2, ..., k$, and $R(\alpha_0) > 0, R(\alpha_{i_1}) > 0$. These conditions are satisfied here since $u_{i_1} < u_{i_2} < \cdots < u_{i_k}$ and $\alpha_0, \alpha_1, ..., \alpha_k$ are parameters of the gamma densities.

Linear combinations such as $a_1 Z_1 + \cdots + a_k Z_k$, where $a_1, ..., a_k$ are constants, can be handled by rewriting them as linear combinations of the mutually independent gamma variables $V_0, V_1, ..., V_k$. Several techniques are available for working out the density of a linear function of independent gamma variables. Quite a large literature is available on this topic. For a brief introduction see Johnson and Kotz [5].

Remarks. (1) In Definition 1, V_0 may be replaced by a sum of independent gamma variables with the same β_0. The structure of the problem remains unchanged. (2). Definition 1 may also be modified to replace $(\beta_j/\beta_0)\ V_0$ by a linear function of independent gamma random variables. The coefficients of the linear function may be selected so that Z_j still remains a gamma. This will lead to a form of the joint distribution of $Z_1, ..., Z_k$ that is more complicated, but it can also be handled by the techniques of this paper.

6. APPROXIMATIONS AND ASYMPTOTIC RESULTS

The m.g.f. of Z_i is given by

$$M_{z_i}(t_i) = E \exp\left\{ t_i \left(\frac{\beta_i}{\beta_0} V_0 + V_i \right) \right\} = e^{t_i(\gamma_0(\beta_i/\beta_0) + \gamma_i)} (1 - \beta_i t_i)^{-(\alpha_0 + \alpha_i)}.$$

Consider the standardized Z_i, that is,

$$Y_i=\frac{Z_i-E(Z_i)}{\sqrt{\operatorname{Var}(Z_i)}}=\frac{Z_i-[(\alpha_0+\alpha_i)\,\beta_i+(\gamma_0/\beta_0)\,\beta_i+\gamma_i]}{\beta_i\sqrt{(\alpha_0+\alpha_i)}}$$

$$=\frac{Z_i}{\beta_i\sqrt{(\alpha_0+\alpha_i)}}-\left[\sqrt{(\alpha_0+\alpha_i)}+\frac{\gamma_0}{\beta_0\sqrt{(\alpha_0+\alpha_i)}}+\frac{\gamma_i}{\beta_i\sqrt{(\alpha_0+\alpha_i)}}\right]. \tag{14}$$

The m.g.f. of Y_i is

$$M_{Y_i}(t_i)=e^{-t_i\sqrt{(\alpha_0+\alpha_i)}}\left(1-\frac{t_i}{\sqrt{(\alpha_0+\alpha_i)}}\right)^{-(\alpha_0+\alpha_i)}$$

and

$$\ln M_{Y_i}(t_i)=\frac{t_i^2}{2}+o\left(\frac{1}{\sqrt{(\alpha_0+\alpha_i)}}\right) \qquad \text{as} \quad \alpha_i\to\infty.$$

Thus we have the following

LEMMA 6.1. *When $\alpha_i\to\infty$ the standardized Z_i as defined in* (14) *is asymptoytically a standard normal irrespective of the value of $\alpha_0>0$.*

Consider the vector of standardized Z_i's, that is, let $\mathbf{Y}'=(Y_1, ..., Y_k)$, where Y_i is defined in (14). The m.g.f. of $\mathbf{Y}$ is then

$$M_Y(t_1, ..., t_k)=E\exp\left\{\sum_{i=1}^{k}\frac{t_iZ_i}{\beta_i\sqrt{(\alpha_0+\alpha_i)}}\right.$$

$$\left.-\sum_{i=1}^{k}t_i\left[\sqrt{(\alpha_0+\alpha_i)}+\frac{\gamma_0}{\beta_0\sqrt{(\alpha_0+\alpha_i)}}+\frac{\gamma_i}{\beta_i\sqrt{(\alpha_0+\alpha_i)}}\right]\right\}$$

$$=e^{-\sum_{i=1}^{k}t_i\sqrt{(\alpha_0+\alpha_i)}}\left[1-\sum_{i=1}^{k}\frac{t_i}{\sqrt{(\alpha_0+\alpha_i)}}\right]^{-\alpha_0}$$

$$\times\prod_{i=1}^{k}\left(1-\frac{t_i}{\sqrt{(\alpha_0+\alpha_i)}}\right)^{-\alpha_i} \tag{15}$$

and without loss of generality one can expand $\ln M_{\mathbf{Y}}(t_1, ..., t_k)$ into a power series to obtain

$$\ln M_{\mathbf{Y}}(t_1, ... t_k)=-\sum_{i=1}^{k}t_i\sqrt{(\alpha_0+\alpha_i)}+\alpha_0\sum_{r=1}^{\infty}\frac{1}{r}\left(\sum_{i=1}^{k}\frac{t_i}{\sqrt{(\alpha_0+\alpha_i)}}\right)^r$$

$$+\sum_{i=1}^{k}\alpha_i\left[\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{t_i}{\sqrt{(\alpha_0+\alpha_i)}}\right)^r\right]$$

$$=\alpha_0\sum_{r=2}^{\infty}\frac{1}{r}\left(\sum_{i=1}^{k}\frac{t_i}{\sqrt{(\alpha_0+\alpha_i)}}\right)^r+\sum_{r=2}^{\infty}\frac{1}{r}\left\{\sum_{i=1}^{k}\alpha_i\left(\frac{t_i}{\sqrt{(\alpha_0+\alpha_i)}}\right)^r\right\}. \tag{16}$$

From (16) one may note that $\ln M_{\mathbf{Y}}$ can be approximated in a number of ways. If α_0 is a finite fixed quantity but $\alpha_i \to \infty$, $i=1, ..., k$ then $\ln M_{\mathbf{Y}}(t_1, ..., t_k) \to \frac{1}{2} \sum_{i=1}^{k} t_i^2$. But if α_0 also goes to ∞ such that $\alpha_i/\alpha_0 \to m_i < \infty$, $i=1, ..., k$, then

$$\ln M_{\mathbf{Y}}(t_1, ..., t_k) \to \frac{1}{2} \left\{ \sum_{i=1}^{k} t_i^2 + \sum_{i \neq j} \frac{t_i t_j}{\sqrt{1+m_i}\sqrt{1+m_j}} \right\}.$$

Thus we have the following theorem.

THEOREM 6.1. *The vector* $\mathbf{Y}$ *of the standardized* Z_i*'s is asymptotically a multivariate normal*

(i) $N_k(\mathbf{0}, \mathbf{I})$ *if* $\alpha_i \to \infty$, $i=1, ..., k$ *and* α_0 *is a fixed finite quantity*;

(ii) $N_k(\mathbf{0}, \mathbf{\Omega})$, $\Omega = (\Omega_{ij})$, $\Omega_{ii} = 1$, $\Omega_{ij} = 1/\sqrt{1+m_i}\sqrt{1+m_j}$, $i \neq j$ *if* $\alpha_i \to \infty$, $i=0, 1, ..., k$ *such that* $\alpha_i/\alpha_0 \to m_i < \infty$, $i=1, ..., k$.

Thus when α_i, $i=1, ..., k$, are large and α_0 is finite, one can approximate the density of $\mathbf{Y}$ by a standardized multinormal density. When $\alpha_i = 0, 1, ..., k$ are large one can approximate by a multinormal density $N_k(\mathbf{0}, \mathbf{\Omega})$.

A situation in Theorem 6.1(ii) can arise when a simple random sample of size n is taken from $\mathbf{Z}' = (Z_1, ..., Z_k)$ and then the standardized sample mean vector of the coordinates is taken as $\mathbf{Y}$. Then look at the distribution of $\mathbf{Y}$ for $n \to \infty$ with α_i, $i=0, 1, ..., k$, fixed. A situation in Theorem 6.1(i) can arise if we take a simple random sample of size n_i from V_i, $i=1, ..., k$, and replace Z_i of Definition 1 by Z_i^*, where

$$Z_i^* = \frac{\beta_i}{n_i \beta_0} V_0 + \bar{V}_i,$$

where $\bar{V}_i$ is the sample mean of the sample from V_i. Now form the vector $\mathbf{Y}$ from Z_i^*, $i=1, ..., k$, and look at the asymptotic distribution of $\mathbf{Y}$ for $n_i \to \infty$, $i=1, ..., k$, with α_i, $i=0, ..., k$, fixed. Thus Lemma 6.1 and Theorem 6.1 can be given meaningful interpretations.

7. ESTIMATION OF PARAMETERS

If data are available on the component gamma variables $V_1, ..., V_k$ then all the parameters $\alpha_i, \beta_i, \gamma_i$, $i=0, ..., k$, can be estimated by using one of the standard methods. Since gamma functions are involved it would be easier to get the estimates by the method of moments. But when a multivariate model is fitted usually data are available only on the coordinates

Z_i, $i=1, \ldots, k$. Hence we will look for a method of estimating the parameters by using data on the Z_i's. Let $m_1^{(i)}$, $m_2^{(i)}$, $m_3^{(i)}$, $m_4^{(i)}$ denote the sample cumulants on Z_i, $i=1, \ldots, k$. The corresponding population cumulants are available from (6). Let S_{ij} denote the sample covariance between Z_i and Z_j. By the method of moments one has

$$m_2^{(i)}=(\hat{\alpha}_0+\hat{\alpha}_i)\,\hat{\beta}_i^2 \tag{17}$$

$$m_3^{(i)}=2(\hat{\alpha}_0+\hat{\alpha}_i)\,\hat{\beta}_i^3, \tag{18}$$

where ^ denotes the estimated value. From (17) and (18),

$$\hat{\beta}_i=b_i, \qquad b_i=\frac{m_3^{(i)}}{2m_2^{(i)}}, \qquad i=1, \ldots, k, \tag{19}$$

and

$$\hat{\alpha}_0+\hat{\alpha}_i=a_i, \qquad a_i=\frac{m_2^{(i)}}{b_i^2}, \qquad i=1, \ldots, k. \tag{20}$$

From the relation $\mathrm{Cov}(Z_1, Z_2)=\alpha_0\beta_1\beta_2$ one has

$$\hat{\alpha}_0=S_{12}/b_1b_2. \tag{21}$$

Any other sample covariance could also be used to estimate α_0. Then

$$\hat{\alpha}_i=a_i-\frac{S_{12}}{b_1b_2}, \qquad i=1, \ldots, k. \tag{22}$$

Note that

$$E(Z_i)=(\alpha_0+\alpha_i)\,\beta_i+\frac{\gamma_0}{\beta_0}\,\beta_i+\gamma_i$$

and hence

$$\delta_0 b_i+\hat{\gamma}_i=m_1^{(i)}-a_ib_i, \qquad \delta_0=\gamma_0/\beta_0. \tag{23}$$

Note that $\delta_0\beta_i+\gamma_i$ is the location parameter for Z_i and hence the smallest order statistic from Z_i, denoted by $Z_{(1)}^{(i)}$, can also estimate $\delta_0\beta_i+\gamma_i$. Also one can get a different moment estimate for β_i by combining the third and fourth sample cumulants. Denoting this estimate by $\tilde{\beta}_i$ one has

$$\tilde{\beta}_i=m_4^{(i)}/3m_3^{(i)}.$$

It should be noted that with probability one, $\tilde{\beta}_i\neq b_i$. If $\tilde{\beta}_i=b_i$ for a given sample, combine higher cumulants to get an estimate of β_i different from b_i. Thus we can have

$$\delta_0\tilde{\beta}_i+\hat{\gamma}_i=Z_{(1)}^{(i)}. \tag{24}$$

From (23) and (24) one has

$$\hat{\delta}_0 = (Z_{(1)}^{(i)} - m_1^{(i)} + a_i b_i)/(\bar{\beta}_i - b_i), \qquad \hat{\gamma}_i = Z_{(1)}^{(i)} - \hat{\delta}_0 \bar{\beta}_i. \tag{25}$$

Since γ_0/β_0 appears as a ratio in the multivariate model the above equations (17)–(25), complete the estimation of all parameters.

8. CHEBYSHEV'S TYPE INEQUALITIES

Since the components Z_i, $i = 1, ..., k$, of our multivariate gamma model are positive random variables for $\gamma_i \geqslant 0$, $i = 0, 1, ..., k$, one can construct Chebyshev type inequalities on various probability contents under our model. The exact probabilities can be computed by using the exact density given in Section 5. We state the following known results as lemmas.

LEMMA 8.1. *Let* $A_1, ..., A_k$ *be events in the same sample space. Let* A_j^c *denote the complement of* A_j *and* $P(A_j)$ *the probability of the event* $A_j, j = 1, ..., k$. *Then*

$$P\left\{\left(\bigcap_{j=1}^{k} A_j\right)\right\} = 1 - P\left\{\left(\bigcup_{j=1}^{k} A_j^c\right)\right\} \geqslant 1 - \sum_{j=1}^{k} P(A_j^c). \tag{26}$$

LEMMA 8.2. *Let* U *be a nonnegative real scalar random variable with* $E(U) < \infty$. *Then*

$$P\{U \geqslant \delta\} \leqslant E(U)/\delta, \qquad \delta > 0. \tag{27}$$

Now consider the events $A_j = \{Z_j \leqslant \delta_j, \delta_j > 0\}$, $j = 1, ..., k$. By applying Lemmas 8.1 and 8.2 and using the expression for the expected value of Z_j one has the following.

THEOREM 8.1. *Let* $\delta_j > 0, j = 1, ..., k$, *be arbitrary real constants. Let* $\{Z_j, j = 1, ..., k\}$ *be the components of the multivariate gamma model of Section* 2, *with* $\gamma_j \geqslant 0, j = 0, 1, ..., k$. *Then*

$$\text{(i)} \quad P\{(Z_1 \leqslant \delta_1, ..., Z_k \leqslant \delta_k)\} \geqslant 1 - \sum_{j=1}^{k} \left\{\left[(\alpha_0 + \alpha_j)\,\beta_j + \frac{\gamma_0}{\beta_0}\,\beta_j + \gamma_j\right]\Big/\delta_j\right\} \tag{28}$$

$$\text{(ii)} \quad P\{(Z_1 \geqslant \delta_1, ..., Z_k \geqslant \delta_k)\} \leqslant \frac{\sum_{j=1}^{k} \{[(\alpha_0 + \alpha_j)\,\beta_j + (\gamma_0/\beta_0)\,\beta_j + \gamma_j]}{\delta_1 + \cdots + \delta_k}. \tag{29}$$

Proof. (i) is obvious from Lemmas 8.1 and 8.2 by noting that $E(Z_j) = (\alpha_0 + \alpha_j)\beta_j + (\gamma_0/\beta_0)\beta_j + \gamma_j$. For proving (ii) observe the following: For nonnegative real scalar random variables $U_1, ..., U_k$ the event $\{U_i \geqslant \delta_i, \delta_i \geqslant 0, i = 1, ..., k\}$ implies that $\{\sum_{i=1}^k U_i \geqslant \sum_{i=1}^k \delta_i\}$. Now apply Lemma 8.2 to establish the result.

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