A FORM OF MULTIVARIATE GAMMA DISTRIBUTION

A. M. Mathai\textsuperscript{1} and P. G. Moschopoulos\textsuperscript{2}

\textsuperscript{1}Department of Mathematics and Statistics, McGill University, Montreal, Canada H3A 2K6
\textsuperscript{2}Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968-0514, U.S.A.

(Received July 30, 1990; revised February 14, 1991)

Abstract. Let $V_i$, $i = 1, \ldots, k$, be independent gamma random variables with shape $\alpha_i$, scale $\beta_i$, and location parameter $\gamma_i$, and consider the partial sums $Z_1 = V_1$, $Z_2 = V_1 + V_2$, \ldots, $Z_k = V_1 + \cdots + V_k$. When the scale parameters are all equal, each partial sum is again distributed as gamma, and hence the joint distribution of the partial sums may be called a multivariate gamma. This distribution, whose marginals are positively correlated has several interesting properties and has potential applications in stochastic processes and reliability. In this paper we study this distribution as a multivariate extension of the three-parameter gamma and give several properties that relate to ratios and conditional distributions of partial sums. The general density, as well as special cases are considered.

Key words and phrases: Multivariate gamma model, cumulative sums, moments, cumulants, multiple correlation, exact density, conditional density.

1. Introduction

The three-parameter gamma with the density

$$f(x; \alpha, \beta, \gamma) = \frac{(x - \gamma)^{\alpha-1} \exp\left(-\frac{x - \gamma}{\beta}\right)}{\beta^{\alpha} \Gamma(\alpha)} , \quad x > \gamma, \quad \alpha > 0, \quad \beta > 0$$

stands central in the multivariate gamma distribution of this paper.

Multivariate extensions of gamma distributions such that all the marginals are again gamma are the most common in the literature. Such extensions involve the standard gamma ($\beta = 1, \gamma = 0$), or the exponential ($\alpha = 1$), see Johnson and Kotz (1972). Other extensions include the multivariate chi-square (Miller et al. (1958), Krishnaiah and Rao (1961) and Krishnaiah et al. (1963)), while the particular case of a bivariate gamma received special consideration, see Kibble (1941), Eagleson (1964), Ghirtis (1967), Moran (1967, 1969) and Sarmanov (1970). The bivariate gamma is particularly useful in modeling the lifetimes of two parallel systems,
see Freund (1961), Becker and Roux (1981), Lingappaiah (1984) and Steel and le Roux (1987). Other applications include uses of the distribution in rainmaking experiments (Moran (1970)).

In this paper we consider a new form of a multivariate gamma that has potential applications in situations in which partial sums of independent, positive random variables are of interest. Such situations appear in the area of reliability and stochastic processes. Let \( V_i, i = 1, \ldots, k \) be the times between successive occurrences of a phenomenon, and let \( Z_i = Z_{i-1} + V_i, i = 1, \ldots, k \), with \( Z_0 = 0 \). Then, \( Z_i \) is the total time required until the \( i \)-th occurrence. In stochastic processes, it is usually assumed that the times \( V_i \) are also identically distributed. In this case, the occurrence time \( Z_i, i \in N \) can be viewed as a renewal process, and the times \( V_i \) can be called renewal times, see for example Cinlar (1975). Note that \( Z_i \) is the \( i \)-th partial sum of the \( V_i \)'s. In actual practice the \( V_i \)'s may be times between arrivals, or, for example, time delays of an airplane at several airports. Then, \( Z_i \) is the total waiting time for the \( i \)-th occurrence, or the total delay at the \( i \)-th airport.

As another example, consider the following application from reliability. An item is installed at time \( Z_0 = 0 \) and when it fails, it is replaced by an identical (or different) item. Then, when the new item fails it is replaced again by another item and the process continues. In this case \( Z_i = Z_{i-1} + V_i \) where \( V_i \) is the time of operation of the \( i \)-th item, and \( Z_i \) is the time at which the \( i \)-th replacement is needed. \( Z_k \) denotes the time interval in which a total of \( k \) items need replacement.

The applications mentioned above, motivate the new form of a multivariate gamma that we consider in this paper. This is given in the form of a theorem.

**Theorem 1.1.** Suppose \( V_1, \ldots, V_k \) are mutually independent where \( V_i \sim G(\alpha_i, \beta, \gamma_i), i = 1, \ldots, k \) (same \( \beta \)). Let

\[
Z_1 = V_1, Z_2 = V_1 + V_2, \ldots, Z_k = V_1 + \cdots + V_k.
\]

Then, the joint distribution of \( Z = (Z_1, \ldots, Z_k)' \) is a multivariate gamma with density function

\[
f(z_1, \ldots, z_k) = \frac{(z_1 - \gamma_1)^{\alpha_1 - 1}}{\beta^{\alpha_1} \prod_{i=1}^{k} \Gamma(\alpha_i)} \cdot \frac{(z_2 - z_1 - \gamma_2)^{\alpha_2 - 1} \cdots (z_k - z_{k-1} - \gamma_k)^{\alpha_k - 1}}{e^{-(z_k - (\gamma_1 + \cdots + \gamma_k))/\beta}}
\]

for \( \alpha_i > 0, \beta > 0, \gamma_i \) real, \( z_i - 1 + \gamma_i < z_i, i = 2, \ldots, k, z_k < \infty, \gamma_1 < z_1, \alpha_k^* = \alpha_1 + \cdots + \alpha_k, \) and zero elsewhere.

Many applications exist in which the \( V_i \)'s represent independent and identically distributed times. Here, they are assumed to be distributed as in (1.1), thus allowing maximum flexibility in shape, scale and location.

The requirement of equal scale parameter \( \beta \) is to ensure that the marginals are of the same form. Basic properties of the distribution of \( Z \), including the moment
A FORM OF MULTIVARIATE GAMMA DISTRIBUTION

generating function, means, variances, properties of the covariance matrix and the reproductive property are given in Section 2. In Section 3 we give the moments and cumulants, and in Section 4 we discuss conditional distributions and special cases.

Before discussing the properties of the model in (1.2) a brief description of the various methods of construction of multivariate gamma distributions will be given here.

After giving a brief sketch of the historical development Dussauchoy and Berland (1974) define a multivariate gamma random variable \( Z = (Z_1, \ldots, Z_n)' \) in terms of the characteristic function defined by

\[
\psi_z(u_1, \ldots, u_n) = \prod_{j=1}^n \frac{\psi_{z_j}(u_j + \sum_{k=j+1}^n \beta_{jk} u_k)}{\psi_{z_j}(\sum_{k=j+1}^n \beta_{jk} u_k)}
\]

where

\[
\psi_{z_j}(u_j) = (1 -iu_j/\alpha_j)^{-\alpha_j}, \quad j = 1, \ldots, n, \quad i = \sqrt{-1}, \quad \beta_{jk} \geq 0, \\
\alpha_j \geq \beta_j \alpha_k > 0, \quad j < k = 1, \ldots, n, \quad 0 < e_1 \leq e_2 \leq \cdots \leq e_n.
\]

Various properties are studied with the help of (1.3) but explicit form of the density is not evaluated except for the bivariate case.

Gaver (1970) considered a mixture of gamma variables with negative binomial weights and came up with a multivariate vector \( Z = (Z_1, \ldots, Z_m)' \) as that one with the Laplace transform of the density given by

\[
L_z(s_1, \ldots, s_m) = \left\{ \frac{\alpha}{(1 + \alpha) \prod_{j=1}^m (s_j + 1) - 1} \right\}^k
\]

for \( k > 0, \alpha > 0 \). He also looked at the possibility of generating a multivariate gamma as a mixture with Poisson weights.

Kowalczyk and Tyrcha (1989) start with the three-parameter gamma in (1.1), denoting the random variable by \( \Gamma(\alpha, \beta, \gamma) \). They call the joint distribution of \( Z_1 = [\sigma_1(V_0 + V_i - \alpha_1)/\sqrt{\alpha_1}] + \mu_i, \quad i = 1, \ldots, k \) as the multivariate gamma, where \( V_0 = \Gamma(\theta_0, 1, 0), \quad V_i = \Gamma(\alpha_i - \theta_0, 1, 0), \quad 0 \leq \theta_0 \leq \min(\alpha_1, \ldots, \alpha_k), \quad \sigma_i > 0, \quad \mu_i \) a real number, \( i = 1, \ldots, k \) and \( V_0, V_1, \ldots, V_k \) are assumed to be mutually independently distributed. They look at some properties including convergence to a multivariate normal and estimation problems.

Mathai and Moschopoulos (1991) start with (1.1) and look at the joint distribution of \( Z_i = (\beta_i/\beta_0) V_0 + V_i \), where \( V_i, \quad i = 0, \ldots, k \) are mutually independently distributed as in (1.1) with different parameters. They look at the explicit form of the multivariate density and study various properties.

None of the multivariate gamma models discussed above or the ones studied by others falls in the category of (1.2) defined for the present study. Hence, we will look at some properties of (1.2) here.
2. Properties

Several properties of the distribution can be obtained from the definition, while others will follow from the moment generating function (m.g.f.). The m.g.f. of \( V_i \) is

\[
M_{V_i}(t) = E(e^{t V_i})
= \beta^{\alpha_i} \Gamma(\alpha_i)^{-1} \int_0^\infty (v_i - \gamma_i)^{\alpha_i - 1} \exp \left( -\frac{v_i - \gamma_i}{\beta} \right) \exp(t v_i) dv_i
= \frac{e^{\gamma_i t}}{(1 - \beta t)^{\alpha_i}}.
\]

From this we get the m.g.f. of \( Z \) as follows.

\[
M_{Z}(t) = M_{Z}(t_1, \ldots, t_k) = E(e^{t_1 Z_1 + \cdots + t_k Z_k})
= \frac{e^{\gamma_1 t_1 + \cdots + t_k}}{(1 - \beta(t_1 + \cdots + t_k))^{\alpha_1}} \frac{e^{\gamma_2 t_2 + \cdots + t_k}}{(1 - \beta(t_2 + \cdots + t_k))^{\alpha_2}} \cdots \frac{e^{\gamma_k t_k}}{(1 - \beta t_k)^{\alpha_k}}.
\]

The m.g.f. exists if \(|t_i + t_{i+1} + \cdots + t_k| < 1/\beta \) for \( i = 1, \ldots, k \). From the definition directly or from the m.g.f. above we obtain the following properties:

(i) The marginal distribution of \( Z_i \) is gamma,

\[
Z_i \sim G(\alpha_i^*, \beta, \gamma_i^*), \quad i = 1, \ldots, k
\]

where \( \alpha_i^* = \alpha_1 + \cdots + \alpha_i \), \( \gamma_i^* = \gamma_1 + \cdots + \gamma_i \).

(ii) The mean and variance of \( Z_i \) are given by

\[
E(Z_i) = \beta \alpha_i^* + \gamma_i^*,
\]

\[
\text{Var}(Z_i) = \beta^2 \alpha_i^*.
\]

(iii) \( Z_i \) and \( Z_j \) are correlated. For \( i < j \) we have

\[
\text{Cov}(Z_i, Z_j) = \text{Cov}(Z_i, Z_i + V_{i+1} + \cdots + V_j) = \text{Var}(Z_i) = \beta^2 \alpha_i^*,
\]

\[
\rho = \text{Corr}(Z_i, Z_j) = \sqrt{\frac{\alpha_i^*}{\alpha_j^*}}.
\]

Clearly, the correlation is always positive. Now, the covariance matrix of \( Z \) is given by

\[
\Sigma = \begin{pmatrix}
\sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 \\
\sigma_1^2 & \sigma_1^2 + \sigma_2^2 & \cdots & \sigma_1^2 + \sigma_2^2 \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_1^2 & \sigma_1^2 + \sigma_2^2 & \cdots & \sigma_1^2 + \cdots + \sigma_k^2
\end{pmatrix},
\]

where \( \sigma_i^2 = \alpha_i \beta^2 \). A matrix of the above structure has several interesting properties that hold regardless of the distribution of the \( V_i \)'s. The determinant of \( \Sigma \) is the product of \( \sigma_i^2, \ i = 1, \ldots, k, \ i.e. \)
(iv)

\(|\Sigma| = \sigma_1^2 \sigma_2^2 \cdots \sigma_k^2.\)

This is easily seen by adding \((-1)\) times the first row to all other rows, then \((-1)\) times the second row to all the following rows etc. It should be noted that the eigenvalues of \(\Sigma\) are not equal to \(\sigma_i^2, i = 1, \ldots, k\). Now if we let \(||\Sigma|| = \max_j \sum_{i=1}^k |\sigma_{ij}|\), where \(\sigma_{ij}\) denotes the \((ij)\)-th element of \(\Sigma\), then we have:

(v)

\(||\Sigma|| = \text{tr}(\Sigma).\)

In general \(||A||\) need not be equal to the trace of \(A\). Next, consider the determinants of the principal minors, starting from the last. These are as follows:

1. \(\Sigma|_{kk} = \sigma_1^2 + \cdots + \sigma_k^2,\)
2. \(\Sigma|_{k-1,k-1} = \det \begin{pmatrix}
\sigma_1^2 + \cdots + \sigma_{k-1}^2 & \sigma_1^2 + \cdots + \sigma_{k-1}^2 \\
\sigma_1^2 + \cdots + \sigma_{k-1}^2 & \sigma_1^2 + \cdots + \sigma_{k-1}^2
\end{pmatrix} = (\sigma_1^2 + \cdots + \sigma_{k-1}^2)\sigma_k^2,\)
3. \(\Sigma|_{22} = (\sigma_1^2 + \sigma_2^2)\sigma_3^2 \cdots \sigma_k^2,\)
4. \(\Sigma|_{11} = \Sigma = \sigma_1^2 \sigma_2^2 \cdots \sigma_k^2.\)

Now, consider the partitioning

\(\Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}, \quad \text{where} \quad \Sigma_{11} = \sigma_1^2 = \sigma_1^2.\)

Then,

\(|\Sigma| = |\Sigma_{22}| \left[ \sigma_1^2 - \sigma_1^2(1, \ldots, 1)\Sigma_{22}^{-1} \begin{pmatrix}
1 \\
1
\end{pmatrix} \right] \quad \text{or,} \quad \sigma_1^2 \sigma_2^2 \cdots \sigma_k^2 = (\sigma_1^2 + \sigma_2^2)\sigma_3^2 \cdots \sigma_k^2 \sigma_1^2(1 - \sigma_2^2\Sigma_{22}^{-1}1)\Sigma_{22}^{-1}1\right] \quad \text{or,}

\(\Sigma_{22}^{-1}1 = \frac{1}{\sigma_1^2 + \sigma_2^2}.\)

Note that the sum of the elements of \(\Sigma_{22}^{-1}\) is free of \(\sigma_3^2, \ldots, \sigma_k^2.\)

From (vi) one can also get the multiple correlation of \(Z_1\) on \(Z_2, \ldots, Z_k\) in a nice form. Using the standard notation

\[R_1^2(2, \ldots, k) = \frac{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}{\sigma_{11}}, \quad \sigma_{11} = \sigma_1^2, \quad \Sigma_{12} = \sigma_1^2(1, \ldots, 1),\]

one has from (vi)
\( R^2_{1(2..k)} = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}. \)

The partial correlation between \( Z_1 \) and \( Z_2 \) given \( Z_3, \ldots, Z_k \), denoted by \( \rho_{12.(3..k)} \), can be seen to have a nice form. Using standard notations

\[
\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \Sigma_{13} \\
\Sigma_{21} & \sigma_{22} & \Sigma_{23} \\
\Sigma_{31} & \Sigma_{32} & \Sigma_{33}
\end{pmatrix},
\]

\[
\rho^2_{12.(3..k)} = \frac{(\sigma_{12} - \Sigma_{13}^{-1} \Sigma_{33}^{-1} \Sigma_{32} \Sigma_{33}^{-1})^2}{(\sigma_{11} - \Sigma_{13}^{-1} \Sigma_{33}^{-1} \Sigma_{31})(\sigma_{22} - \Sigma_{23} \Sigma_{33}^{-1} \Sigma_{32})}
\]

where \( \sigma_{11} = \sigma_1^2 \), \( \sigma_{22} = \sigma_2^2 \), \( \sigma_{22} - \Sigma_{23} \Sigma_{33}^{-1} \Sigma_{32} = (\sigma_2^2 + \sigma_3^2)[1 - (\sigma_1^2 + \sigma_2^2) \Sigma_{33}^{-1} \Sigma_{33}^{-1}] \);

\[
\sigma_{11} = \Sigma_{13} \Sigma_{33}^{-1} \Sigma_{31} = \sigma_1^2 [1 - \sigma_1^2 \Sigma_{33}^{-1} \Sigma_{33}^{-1}],
\]

\[
\sigma_{22} - \Sigma_{23} \Sigma_{33}^{-1} \Sigma_{32} = (\sigma_2^2 + \sigma_3^2)[1 - (\sigma_1^2 + \sigma_2^2) \Sigma_{33}^{-1} \Sigma_{33}^{-1}],
\]

But it is easy to note that \( \Sigma_{33}^{-1} \Sigma_{33}^{-1} = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)^{-1} \). Thus one has

\(
\rho^2_{12.(3..k)} = \frac{\sigma_1^2 \sigma_2^2}{(\sigma_1^2 + \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}.
\)

(ix) (Reproducitive property) Let \( W_1 \) be a multivariate gamma with parameters \( \alpha_i, \beta, \gamma_i, i = 1, \ldots, k \), and \( W_2 \) independently distributed as a multivariate gamma with parameters \( \alpha_i', \beta, \gamma_i', i = 1, \ldots, k \). Then, it is clear from the m.g.f. in (2.1) that \( W_1 + W_2 \) is also distributed as a multivariate gamma with parameters \( \alpha_i + \alpha_i', \beta, \gamma_i + \gamma_i', i = 1, \ldots, k \).

3. Moments and cumulants

The cumulant generating function of \( Z \) is the logarithm of the m.g.f. in (2.2) and is given by

\[(3.1) \quad K_z(t) = \gamma_1 \sum_{i=1}^{k} t_i + \gamma_2 \sum_{i=2}^{k} t_i + \cdots + \gamma_k t_k
\]

\[\quad - \alpha_1 \ln[1 - \beta \sum_{i=1}^{k} t_i] - \alpha_2 \ln[1 - \beta \sum_{i=2}^{k} t_i] - \alpha_k \ln[1 - \beta t_k].\]

Thus, the \( m \)-th cumulant of \( Z_i \) and the \((m_1, m_2)\)-th product cumulant of \( Z_i \) and \( Z_j \) are given by

\[(3.2) \quad K_m = \frac{\partial^m}{\partial t_1^m} \ln M_z(t)\big|_{t=0} = \begin{cases} \gamma_i + \beta \alpha_i^*, & \text{if } m = 1 \\
(m-1)! \beta^m \alpha_i^*, & \text{if } m \geq 2,\end{cases}
\]

\[(3.3) \quad K_{m_1, m_2} = \frac{\partial^{m_1+m_2}}{\partial t_1^{m_1} \partial t_2^{m_2}} \ln M_z(t)\big|_{t=0} = (m_1 + m_2 - 1)! \beta^{m_1+m_2} \alpha_i^*, \]

where \( r = \min(i, j) \). Next, we obtain the moments of \( Z_i \). These are easier to get from the moments of \( V_i \).

\[
\frac{d^m M_{v_i}(t)}{dt^m} = \sum_{k_1=0}^{m} \binom{m}{k_1} \left( \frac{d^{k_1}}{dt^{k_1}} (1 - t)^{-\alpha_i} \right) \left( \frac{d^{m-k_1}}{dt^{m-k_1}} e^{(\gamma_i t)} \right).
\]

Hence, putting \( t = 0 \) we get the \( m \)-th moment of \( V_i \):

\[
M_{i}^{(m)} = E(V_i^m) = \sum_{k_1=0}^{m} \binom{m}{k_1} (\alpha_i + 1)^{k_1} (\alpha_i + k_1 - 1)\beta_i^{k_1} \gamma_i^{m-k_1}
\]

\[
= \sum_{k_1=0}^{m} \binom{m}{k_1} \alpha_i^{k_1} \beta_i^{k_1} \gamma_i^{m-k_1}
\]

where \( (\alpha)_r = \alpha(\alpha + 1) \cdots (\alpha + r - 1) \), \( (\alpha)_0 = 1 \). Using the above, we now have,

\[
E(Z_i^m) = E(V_1 + \cdots + V_i)^m = \sum_{\kappa(r_1, \ldots, r_k, m)} \frac{m!}{r_1! \cdots r_i!} \prod_{i=1}^{k} E(V_i^{r_i})
\]

\[
= \sum_{\kappa(r_1, \ldots, r_k, m)} \frac{m!}{r_1! \cdots r_i!} \prod_{i=1}^{k} \{M_{i}^{(r_i)}\}
\]

where \( M_{i}^{(r_i)} \) is given in (3.4),

\[
\kappa(r_1, \ldots, r_k, m) = \{ (r_1, \ldots, r_k) \in N_+^k \mid r_1 + \cdots + r_k = m \}
\]

and \( N_+ \) is the set of non-negative integers. Similarly,

\[
E(Z_i^{\beta_1} Z_j^{\beta_2}) = \sum_{\kappa(r_1, \ldots, r_k)} \sum_{\kappa(s_1, \ldots, s_j)} \frac{k_1!}{r_1! \cdots r_i!} \frac{k_2!}{s_1! \cdots s_j!} Q
\]

where

\[
Q = \left\{ \begin{array}{ll}
E(V_i)^{r_1+s_1} \cdots E(V_i)^{r_i+s_i} E(V_{i+1})^{s_{i+1}} \cdots E(V_j)^{s_j} & \text{if } j > i \\
E(V_i)^{r_1+s_1} \cdots E(V_j)^{r_{j-1}+s_{j-1}} E(V_{j+1})^{s_{j+1}} \cdots E(V_i)^{s_j} & \text{if } j < i \\
E(Z_i)^{k_1+k_2} & \text{if } j = i.
\end{array} \right.
\]

4. \textbf{Densities}

From the joint distribution in (1.2) we can obtain the distribution of subsets of \( Z_1, \ldots, Z_k \). First, consider the density of \( (Z_1, \ldots, Z_{k-1}) \). This is easily obtained by integrating out \( Z_k \). Integration over \( Z_k \) leads to the following integral

\[
\int_{z_{k-1}+z_k}^{\infty} (z_k - z_{k-1} - \gamma_k)^{\alpha_k-1} e^{-(z_k - (\gamma_1 + \cdots + \gamma_k))}/\beta \, dz_k
\]

\[
= \int_{0}^{\infty} u^{\alpha_k-1} e^{-u + z_{k-1} + \gamma_k - (\gamma_1 + \cdots + \gamma_k)}/\beta \, du
\]

\[
= \beta^{\alpha_k} \Gamma(\alpha_k) e^{-(z_{k-1} - (\gamma_1 + \cdots + \gamma_{k-1}))}/\beta.
\]
Hence, the joint density of \( Z_1, \ldots, Z_{k-1} \) is of the same form as the density of \( Z_1, \ldots, Z_k \). This is also clear from the definition of the \( Z_i \)'s.

Next, consider the joint density of \( Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_k \). This is obtained by integrating out \( Z_i \), which leads to the following integral:

\[
(4.2) \int_{z_{i-1} + \gamma_i}^{z_{i+1} - \gamma_i} (z_i - z_{i-1} - \gamma_i)^{\alpha_i} (z_{i+1} - z_i - \gamma_{i+1} \gamma_i)^{\alpha_{i+1}-1} dz_i
\]

\[
= \int_0^{z_{i+1} - z_{i-1} - \gamma_i} u^{\alpha_i} (z_{i+1} - z_i - \gamma_{i+1} \gamma_i - u)^{\alpha_{i+1}-1} du
\]

\[
= \frac{\Gamma(\alpha_i) \Gamma(\alpha_{i+1})}{\Gamma(\alpha_i + \alpha_{i+1})} (z_{i+1} - z_i - \gamma_{i+1} \gamma_i)^{\alpha_i + \alpha_{i+1}-1}.
\]

Note that the location parameter of \( z_{i+1} - z_{i-1} \) is \( \gamma_1 + \cdots + \gamma_{i+1} - (\gamma_1 + \cdots + \gamma_i) = \gamma_i + \gamma_{i+1} \) and the shape parameter is \( \alpha_i + \alpha_{i+1} \); hence, the joint distribution of the subset \( Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_k \) is also of the same form as the density of the \( Z_1, \ldots, Z_k \).

The above discussion shows that the joint density of all subsets of \( Z_1, \ldots, Z_k \) is of the same functional form. We now establish several interesting results concerning conditional densities and densities of ratios of the \( Z_i \)'s.

(a) The conditional density of \( Z_{i+1} \) given \( Z_i = z_i \) is evidently a gamma with parameters \( \alpha_{i+1}, \beta, z_i + \gamma_{i+1} \), i.e. \( Z_{i+1} \mid Z_i \sim G(\alpha_{i+1}, \beta, z_i + \gamma_{i+1}) \). We note that for \( j > i \) we have

\[
(b) \quad \frac{Z_i - \gamma_i}{Z_j - \gamma_j} \sim \text{Beta}(\alpha_i^*, \alpha_j^* - \alpha_i^*) \quad \text{type-1}.
\]

\[
(c) \quad \frac{Z_i - \gamma_i}{Z_j - Z_i + \gamma_j - \gamma_i} \sim \text{Beta}(\alpha_i^*, \alpha_j^* - \alpha_i^*) \quad \text{type-2}.
\]

Also, since \( (V_i - \gamma_i) / \beta \sim G(\alpha_i, 1, 0) \), as a consequence of a well known result (see, for example Wilks (1962)), we have the following:

\[
(d) \quad Y_1 = \frac{Z_1 - \gamma_1}{Z_k - \gamma_k}, Y_2 = \frac{Z_2 - Z_1 - \gamma_2}{Z_k - \gamma_k}, \ldots, Y_{k-1} = \frac{Z_{k-1} - \gamma_{k-1} - \gamma_k}{Z_k - \gamma_k}
\]

jointly have the Dirichlet density with parameters \( \alpha_1, \ldots, \alpha_k \) and they are independent of \( Z_k \). The Dirichlet density is

\[
h(y_1, y_2, \ldots, y_{k-1}) = \frac{\Gamma(\alpha_k)}{\prod_{i=1}^k \Gamma(\alpha_i)} \left( \prod_{i=1}^{k-1} y_i^{\alpha_i-1} \right) (1 - y_{k-1})^{\alpha_k-1}
\]

where \( y_i^* = y_1 + \cdots + y_i \geq 0, i = 1, \ldots, k - 1 \) and \( \sum_{i=1}^{k-1} y_i \leq 1 \). Clearly, each \( Y_i \) is a Beta type-1 (see (b)).
Finally we note the following results that concern the special case in which each of the \( V_i \)'s is exponential \((\alpha_i = 1, \beta = 1, \gamma_i = 0)\). From the density above we have:

\[
h(y_1, y_2, \ldots, y_{k-1}) = (k - 1)!
\]

and hence \( Y_1, \ldots, Y_{k-1} \) are distributed like the order statistic from the uniform \( U(0,1) \)-distribution. In this case, the joint density of \( Z_1, \ldots, Z_k \) reduces to

\[
f(z_1, \ldots, z_k) = e^{-z_k}, \quad 0 < z_1 < z_2 < \cdots < z_{k-1}, \quad 0 < z_k < \infty.
\]

The transformation

\[
W_1 = \frac{Z_1}{Z_k}, W_2 = \frac{Z_2}{Z_k}, \ldots, W_{k-1} = \frac{Z_{k-1}}{Z_k}, W_k = Z_k
\]

is one-to-one with Jacobian \( J(Z \rightarrow W) = W_k^{k-1} \). Thus, the joint density of \( W_1, \ldots, W_{k-1} \) is

\[
g(w_1, \ldots, w_{k-1}) = \int_{0}^{\infty} w_k^{k-1} e^{-w_k} dw_k = (k - 1)!.\]

Hence, \( W_1, \ldots, W_{k-1} \) are also distributed like the order statistic from the uniform \( U(0,1) \)-distribution.

Acknowledgements

The authors would like to thank the referees for the constructive suggestions which enabled them to improve the manuscript. The first author would like to thank the Natural Sciences and Engineering Research Council of Canada for financial support.

References


