STATISTICA, anno LIX, n. 1, 1999

DISTANCE BETWEEN RANDOM POINTS IN A CUBE

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1. INTRODUCTION

The pioneering work of Ghosh (1943a,b, 1949, 1951) seems to be the first published papers containing explicit expressions for the density functions of the distance between two random points in a square or in a rectangle. In these remarkable contributions the densities are derived for three distinct configurations: a rectangle, two adjacent rectangles and squares having a common diagonal. Ghosh was tackling some problems arising from sample survey studies, a study undertaken at the Indian Statistical Institute under the direction of Professor P.C. Mahalanobis. The author was mainly interested in topographic variation in statistical fields.

Sample survey is not the only field where one requires the statistical properties of a straight path across a preassigned geometrical figure. Several distance methods have been considered by Persson (1964, 1965) in two studies related to forestry. In physics, some problems of interest are to assess the length of a gamma-ray to the wall of a nuclear reactor from a given source and the distance of a sound-ray in a chamber from one reflection to the next, or the measure of a direct trajectory across a square or a cube. Horowitz (1965) considered such problems in a variety of basic geometric shapes when the radiation sources are assumed to be isotropic. Some results in this direction were obtained much earlier by Jäger (1911). Another interesting application is concerned with the traffic flow in the central area of a town with a rectangular grid as the road system, as reported in Smeed and Jeffcoate (1963). The two survey articles by Moran (1966, 1969) also contain a list of applications, and so does the book by Eilon, Watson-Gandy and Christofides (1971). Random paths across elementary geometrical shapes are also considered by Kendall and Moran (1963), Coleman (1969) and Kingman (1965, 1969).

In the present article we consider the distance between two points which are independently and uniformly distributed inside a cube of side a. A special case of this distance also describes the distance between two independent random points in a square, a case considered by Ghosh, as mentioned earlier. Furthermore, explicit forms of the exact density and arbitrary moments of the distance are evaluated in terms of Gauss' hypergeometric functions and Appell's function F_1 which is also Lauricella function F_D for two variables. Simpler forms are shown to be available for positive integer moments of the squared distance. Special cases of the results also include the corresponding results for the distance between two random points inside a square.

2. THE DISTANCE BETWEEN TWO RANDOM POINTS INSIDE A CUBE

Consider a cube of a side a and let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be two random points inside this cube. Randomness means that x_j, y_j, z_j, $j = 1, 2$ are mutually independently and uniformly distributed over $[0, a]$. The distance between the points P and Q is then

$$x = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\ .$$

One method of deriving the density of x is to look at the density of the sum $u = x^2 = u_1 + u_2 + u_3$ where

$$u_1 = (x_2 - x_1)^2, \qquad u_2 = (y_2 - y_1)^2, \qquad u_3 = (z_2 - z_1)^2\ .$$

Then the densities of u_1, u_2, u_3 are available from elementary computations. They are

$$f_j(u_j) = \frac{1}{a^2}\left[a u_j^{-\frac{1}{2}} - 1\right], \qquad 0 \le u_j \le a^2, \qquad j = 1,2,3.$$

Making the transformation $u_1 = t$, $u_2 = v_1 - t$ the joint density of v_1 and t is

$$f(v_1, t) = f_1(t) f_2(v_1 - t) = \frac{1}{a^4}\left\{a^2 t^{-\frac{1}{2}}(v_1 - t)^{-\frac{1}{2}} - a t^{-\frac{1}{2}} - a(v_1 - t)^{-\frac{1}{2}} + 1\right\}$$

and the marginal density of v_1, denoted by $g_1(v_1)$, is available as

$$g_1(v_1) = \begin{cases} h_1(v_1), & 0 \le v_1 \le a^2 \\ h_2(v_2), & a^2 \le v_1 \le 2a^2 \end{cases}$$

where

$$h_1(v_1) = \int_0^{v_1} f(v_1, t)\, dt = \frac{1}{a^4}\left\{\pi a^2 - 4 a v_1^{\frac{1}{2}} + v_1\right\}, \qquad 0 \le v_1 \le a^2$$

and

$$h_2(v_1) = \frac{1}{a^4}\left\{-2a^2 - v_1 + 4a(v_1 - a^2)^{\frac{1}{2}}\right.$$

$$\left. + 2a^2\left[\sin^{-1}\frac{a}{\sqrt{v_1}} - \cos^{-1}\frac{a}{\sqrt{v_1}}\right]\right\}, \qquad a^2 \le v_1 \le 2a^2 .$$

Take

$$g_2(v_2) = f_3(v_2) = \frac{1}{a^2}\left\{av_2^{-\frac{1}{2}} - 1\right\}, \qquad 0 \le v_2 \le a^2 .$$

Then $u = x^2 = v_1 + v_2$. That part of the joint density of $w_1 = v_1 + v_2$ and $w = v_1$ for $0 \le v_1 \le a^2$ and $0 \le v_2 \le a^2$ is then given by

$$g_1(w_1, w) = h_1(w)g_2(w_1 - w) = \frac{1}{a^6}\left\{-\pi a^2 + \pi a^3(w_1 - w)^{-\frac{1}{2}} + 4aw^{\frac{1}{2}}\right.$$

$$\left. - 4a^2w^{\frac{1}{2}}(w_1 - w)^{-\frac{1}{2}} - w + aw(w_1 - w)^{-\frac{1}{2}}\right\}.$$

Note that one has to look into the regions in the (w_1, w)-plane corresponding to the regions $S_1 = \{0 \le v_1 \le a^2, 0 \le v_2 \le a^2\}$ and $S_2 = \{a^2 \le v_1 \le 2a^2, 0 \le v_2 \le a^2\}$. For the set S_1 the integral is over $0 \le w \le w_1$ when $0 \le w_1 \le a^2$ and over $w_1 - a^2 \le w \le a^2$ when $a^2 \le w_1 \le 2a^2$. Similarly for the set S_2 the integral is when $a^2 \le w \le w_1$ when $a^2 \le w_1 \le 2a^2$ and over $w_1 - a^2 \le w \le 2a^2$ when $2a^2 \le w_1 \le 3a^2$. Denoting these four integrals by $f_{1.1}(w_1)$, $f_{1.2}(w_1)$, $f_{2.1}(w_1)$ and $f_{2.2}(w_1)$ respectively one has the following where $f_{1.1}$ and $f_{1.2}$ correspond to S_1 and the remaining to S_2:

$$f_{1.1}(w_1) = \int_0^{w_1} g_1(w_1, w)\,dw$$

$$= \frac{1}{a^6}\left\{-3\pi a^2 w_1 + 2\pi a^3 w_1^{\frac{1}{2}} + 4aw_1^{\frac{3}{2}} - \frac{1}{2}w_1^2\right\}, \qquad 0 \le w_1 \le a^2 \tag{2.1}$$

$$f_{1.2}(w_1) = \int_{w_1 - a^2}^{a^2} g_1(w_1, w)\,dw$$

$$= \frac{1}{a^6}\left\{-2a^2 + (\pi a^2 + a^2)w_1 + \frac{1}{2}w_1^2 - 2\pi a^3(w_1 - a^2)^{\frac{1}{2}}\right.$$

$$- 2aw_1(w_1 - a^2)^{\frac{1}{2}} - 2a(w_1 - a^2)^{\frac{3}{2}}$$

$$\left. - 4a^2 w_1\left[\sin^{-1}\frac{a}{\sqrt{w_1}} - \cos^{-1}\frac{a}{\sqrt{w_1}}\right]\right\}, \qquad a^2 \le w_1 \le 2a^2 . \tag{2.2}$$

The remaining ones $f_{2.1}(w_1)$ and $f_{2.2}(w_1)$ are coming from

$$g_2(w_1,w)=h_2(w)g_2(w_1-w), a^2\le v_1\le 2a^2, 0\le v_2\le a^2$$

$$=\frac{1}{a^6}\Big\{2a^2+w-2a^3(w_1-w)^{-\frac{1}{2}}-aw(w_1-w)^{-\frac{1}{2}}$$

$$+4a^2(w-a^2)^{\frac{1}{2}}(w_1-w)^{-\frac{1}{2}}-4a(w-a^2)^{\frac{1}{2}}$$

$$+2a^3(w_1-w)^{-\frac{1}{2}}\left[\sin^{-1}\frac{a}{\sqrt{w}}-\cos^{-1}\frac{a}{\sqrt{w}}\right]$$

$$-2a^2\left[\sin^{-1}\frac{a}{\sqrt{w}}-\cos^{-1}\frac{a}{\sqrt{w}}\right]\Big\}.$$

Then

$$f_{1.2}(w_1)=\int_{a^2}^{w_1}g_2(w_1,w)\,dw$$

$$=\frac{1}{a^6}\Big\{-\pi a^4-\frac{5}{2}a^4+(2\pi a^2+2a^2)w_1+\frac{1}{2}w_1^2$$

$$+(2\pi a^3-8a^3)(w_1-a^2)^{\frac{1}{2}}-2aw_1(w_1-a^2)^{\frac{1}{2}}$$

$$-2a(w_1-a^2)^{-\frac{3}{2}}-2a^2w_1\left[\sin^{-1}\frac{a}{\sqrt{w_1}}-\cos^{-1}\frac{a}{\sqrt{w_1}}\right]$$

$$-8a^3\int_0^{\cos^{-1}\frac{a}{\sqrt{w_1}}}\left[w_1-\frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}}d\theta\Big\} \tag{2.3}$$

and

$$f_{2.2}(w_1)=\frac{1}{a^6}\Big\{-\frac{5}{2}a^4-3a^2w_1-\frac{1}{2}w_1^2+8a^3(w_1-2a^2)^{\frac{1}{2}}$$

$$+2a(w_1-2a^2)^{\frac{3}{2}}+2aw_1(w_1-2a^2)^{\frac{1}{2}}$$

$$+6a^2(w_1-a^2)\left[\sin^{-1}\frac{a}{\sqrt{w_1-a^2}}-\cos^{-1}\frac{a}{\sqrt{w_1-a^2}}\right]$$

$$+4a^2\left[\sin^{-1}\frac{a}{\sqrt{w_1-a^2}}-\cos^{-1}\frac{a}{\sqrt{w_1-a^2}}\right]$$

$$-8a^3\int_{\cos^{-1}\frac{a}{\sqrt{w_1-a^2}}}^{\frac{\pi}{4}}\left[w_1-\frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}}d\theta\Big\}. \tag{2.4}$$

Note that $f_{1.2}+f_{2.1}$ is that part of the density for $a^2 \le w_1 \le 2a^2$. The density of $u = w_1 = x^2$, denoted by $f(u)$, is the following. This will be stated as a theorem.

Theorem 1.

$$f(u)=\begin{cases} f_{1.1}(u), & 0\le u\le a^2 \\ f_{1.2}(u)+f_{2.1}(u), & a^2\le u\le 2a^2 \\ f_{2.2}(u), & 2a^2\le u\le 3a^2 \end{cases}$$

where $f_{1.1}$ is given in (2.1), $f_{2.1}$ in (2.4) and

$$\begin{aligned} f_{1.2}(u)+f_{2.1}(u) &= \frac{1}{a^6}\Big\{-\pi a^4 - \frac{1}{2}a^4 + (3\pi a^2 + 3a^2)u + u^2 \\ &+ 8a^3(u-a^2)^{\frac{3}{2}} - 4au_1(u-a^2)^{\frac{1}{2}} - 4a(u-a^2)^{\frac{3}{2}} \\ &- 6a^2u\left[\sin^{-1}\frac{a}{\sqrt{u}} - \cos^{-1}\frac{a}{\sqrt{u}}\right] \\ &+ 4a^2\left[\sin^{-1}\frac{a}{\sqrt{w_1-a^2}} - \cos^{-1}\frac{a}{\sqrt{w_1-a^2}}\right] \\ &- 8a^3\int_0^{\cos^{-1}\frac{a}{\sqrt{u}}}\left[u-\frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}}d\theta\Big\}. \end{aligned} \tag{2.5}$$

Then the density of the distance $x = u^{1/2}$, denoted by $g(x)$, is given by

$$g(x) = 2xf(x^2)\,. \tag{2.6}$$

In order to show that $f(u)$ as given above is in fact a density one has to evaluate the two integrals appearing in (2.4) and (2.5). Making the transformation $z = a^2/\cos^2\theta$ one has

$$I_1 = -8a^3\int_0^{\cos^{-1}\frac{a}{\sqrt{u}}}\left[u-\frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}}d\theta = -4a^4\int_{a^2}^{u}\frac{(u-z)^{\frac{1}{2}}}{z\sqrt{z-a^2}}dz,\; a^2\le u\le 2a^2 \tag{2.7}$$

and

$$\begin{aligned} I_2 &= -8a^3\int_{\cos^{-1}\frac{a}{\sqrt{u-a^2}}}^{\frac{\pi}{4}}\left[u-\frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}}d\theta \\ &= -4a^4\int_{u-a^2}^{2a^2}\frac{(u-z)^{\frac{1}{2}}}{z\sqrt{z-a^2}}dz,\; 2a^2\le u\le 3a^2\,. \end{aligned} \tag{2.8}$$

Combining the regions $\{a^2 \le u \le 2a^2, a^2 \le z \le u\}$ and $\{2a^2 \le u \le 3a^2, u - a^2 \le z \le 2a^2\}$ and then changing the order of integration the sum of the two integrals can be evaluated. Since the steps in this derivation are going to be used later on, this result will be stated as a lemma.

Lemma 1.

$$I = \int_{a^2}^{2a^2} I_1 du + \int_{2a^2}^{3a^2} I_2 du = -4a^4 \left\{ \int_{z=a^2}^{2a^2} \frac{1}{z\sqrt{z-a^2}} \left[\int_{u=z}^{z+a^2} (u-z)^{\frac{1}{2}} du \right] dz \right\}$$

$$= -\frac{8}{3} a^7 \int_{z=a^2}^{2a^2} \frac{1}{z\sqrt{z-a^2}} dz = -\frac{8}{3} a^7 \int_{z=0}^{a^2} \frac{1}{(z-a^2)\sqrt{z}} dz = -\frac{1}{3}\pi a^6 \qquad (2.9)$$

by making the substitution $z = a^2 \tan^2 \phi$.

The general h-th moment of u_1 that is $E(u^h) = E(x^{2h})$, for an arbitrary h, can be evaluated by computing the integrals

$$\int_0^{a^2} u^h f_{1.1}(u) du, \int_{a^2}^{2a^2} u^h (f_{1.2} + f_{2.1}) du, \int_{2a^2}^{3a^2} u^h f_{2.2}(u) du.$$

In order to achieve this we need some integrals which will be stated as lemmas here. Special cases for $h = 1/2$, 1 are needed if we want to obtain the mean value and variance of x and hence these special cases are also listed with some of the lemmas.

Lemma 2. For $0 < \gamma \le 1$, $\Re(h) > -1$, $\Re(\delta) > -1$, where $\Re(\cdot)$ denotes the real part of $(\cdot)$

$$\int_0^{\gamma} v^{\delta}(1+v)^h dv = \frac{1}{(\delta+1)^2} F_1(-h, \delta+1; \delta+2; -\gamma)$$

$$= \frac{\sqrt{3}}{4} - \frac{1}{4}\ln\sqrt{2} + \frac{1}{4}\ln(\sqrt{3}-1) \text{ for } \delta = \frac{1}{2}, h = \frac{1}{2}, \gamma = \frac{1}{2}$$

$$= \frac{1}{8}\ln\sqrt{2} - \frac{1}{8}\ln(\sqrt{3}-1) \text{ for } \delta = \frac{3}{2}, h = \frac{1}{2}, \gamma = \frac{1}{2}$$

$$= \frac{3}{4}\sqrt{2} - \frac{1}{4}\ln(\sqrt{2}-1) \text{ for } \delta = \frac{1}{2}, h = \frac{1}{2}, \gamma = 1$$

$$= \frac{7}{24}\sqrt{2} - \frac{1}{8}\ln(\sqrt{2}-1) \text{ for } \delta = \frac{3}{2}, h = \frac{1}{2}, \gamma = 1$$

$$= \frac{19\sqrt{2}}{280} \text{ for } \delta = \frac{3}{2}, h = 1, \gamma = 1$$

$$= \frac{13\sqrt{2}}{60} \text{ for } \delta = \frac{1}{2}, h = 1, \gamma = \frac{1}{2}$$

$$= \frac{24}{35} \text{ for } \delta = \frac{3}{2}, h = 1, \gamma = 1$$

$$= \frac{16}{15} \text{ for } \delta = \frac{1}{2}, h = 1, \gamma = 1,$$

where ${}_2F_1(\cdot)$ is a Gauss' hypergeometric function.

The integral is evaluated by expanding $(1+v)^b$ and integrating term by term.

Lemma 3. For $\gamma > 1$, $\Re(\alpha) > -1$

$$\int_0^{\frac{1}{\gamma}} t^{\alpha}(1-t)^{-\beta}(\gamma-t)^{-\delta}\,dt = \frac{\gamma^{-\delta-1}}{(\alpha+1)} F_1\left(\alpha+1,\beta,\delta;\alpha+2;\frac{1}{\gamma},\frac{1}{\gamma^2}\right)$$

where F_1 is an Appell's double hypergeometric function which is also the Lauricella function F_D for two variables, see for example Mathai (1993).

The integral is evaluated by expanding $(1-t)^{-\beta}$ and $\left(1-\frac{t}{\gamma}\right)^{-\delta}$ by using binomial expansions, integrating out and then interpreting the double series.

Lemma 4. For $m = 1, 2, \ldots$

$$\int \frac{1}{(\cos\theta)^{2m}}\,d\theta = \frac{\sin\theta}{(2m-1)}\left\{\frac{1}{(\cos\theta)^{2m-1}} + \frac{2(m-1)}{(2m-3)}\frac{1}{(\cos\theta)^{2m-3}} + \frac{2^2(m-1)(m-2)}{(2m-3)(2m-5)}\frac{1}{(\cos\theta)^{2m-5}} + \ldots + 2^{m-1}\frac{(m-1)(m-2)\ldots 1}{(2m-3)(2m-5)\ldots 1}\frac{1}{\cos\theta}\right\}$$

and

$$\int \frac{1}{(\cos\theta)^{2m+1}}\,d\theta = \frac{\sin\theta}{2m}\left\{\frac{1}{(\cos\theta)^{2m}} + \sum_{k=1}^{m-1}\frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m-1)(m-2)\ldots(m-k)}\frac{1}{(\cos\theta)^{2m-2k}} + \frac{(2m-1)!}{2^m m!}\ln\left(\frac{\cos\theta}{1-\sin\theta}\right)\right\}.$$

The results are available by successive integration, see also Gradshteyn and Ryzhik (1980) formulae 2.519.1 and 2.519.2. But for computing arbitrary moments of x or of $u = x^2$ we need $\mathrm{E}(u^b)$ for an arbitrary b which in turn will require $\int(\cos^2\theta)^{-\alpha}\,d\theta$ for an arbitrary α. This can be by substituting $t = \sin\theta$, expanding $(1-t^2)^{-(\alpha+1/2)}$ with the help of a binomial expansion and then integrating. The final result is stated next.

Lemma 5.

$$\int \frac{1}{(\cos^2\theta)^{\alpha}}\,d\theta = \sin\theta\; {}_2F_1\left(\alpha+\frac{1}{2},\frac{1}{2};\frac{3}{2};\sin^2\theta\right).$$

It can be seen that for $\alpha = 1, 2, \ldots$ this result agrees with those in Lemma 4. In order to see this, use the result

$$ {}_2F_1(a, b; c; x) = (1 - x)^{-b}\, {}_2F_1(c - a, b; c; x/(x - 1)) . $$

With the help of the above lemmas the h-th moment of u, for an arbitrary h, is the following:

Theorem 2. For $\Re(h) > -1$,

$$
\begin{aligned}
\mathrm{E}(u^h) = a^{2h+6}\Big\{ & \frac{b_1}{(h+1)} + \frac{b_2}{(h+2)} + \frac{b_3}{(h+3)} \\
& + \frac{b_4}{(2h+3)} + \frac{b_5}{(2h+4)} + \frac{b_6}{(2h+5)} + \phi_1 + \phi_2 \\
& - \frac{16}{3}\,{}_2F_1\Big(-h, \frac{3}{2}; \frac{5}{2}; -1\Big) - \frac{8}{3}\,{}_2F_1\Big(-h-1, \frac{3}{2}; \frac{5}{2}; -1\Big) \\
& - \frac{8}{5}\,{}_2F_1\Big(-h, \frac{5}{2}; \frac{7}{2}; -1\Big) + \frac{16}{3} 2^h\,{}_2F_1\Big(-h, \frac{3}{2}; \frac{5}{2}; -\frac{1}{2}\Big) \\
& + \frac{4}{5} 2^h\,{}_2F_1\Big(-h, \frac{5}{2}; \frac{7}{2}; -\frac{1}{2}\Big) \\
& + \frac{2^{h+4}\sqrt{2}}{(h+1)} F_1\Big(\frac{1}{2}, h+\frac{3}{2}, -h-1; \frac{3}{2}; \frac{1}{2}, \frac{1}{4}\Big) \\
& + 3\pi 2^{h-1} F_1\Big(1, -h, h+3; 2; \frac{1}{4}, \frac{1}{2}\Big) \\
& + 12\frac{2^{h+1}\sqrt{2}}{(h+2)} F_1\Big(\frac{1}{2}, h+\frac{5}{2}, -h-2; \frac{3}{2}; \frac{1}{2}, \frac{1}{4}\Big) \\
& - 12\frac{2^{h}\sqrt{2}}{(h+1)} F_1\Big(\frac{1}{2}, h+\frac{3}{2}, -h-1; \frac{3}{2}; \frac{1}{2}, \frac{1}{4}\Big)\Big\}
\end{aligned}
$$

where

$$ b_1 = \pi + \frac{1}{2} + (2 - 3\pi)2^{h+1} + \Big(3\pi - \frac{5}{2}\Big)3^{h+1} $$

$$ b_2 = -3(1 + 2\pi) + 3(2 + \pi)2^{h+2} - 3(1 + \pi)3^{h+2} $$

$$ b_3 = -\frac{3}{2} + \frac{3}{2}2^{h+3} - \frac{1}{2}3^{h-3} $$

$$ b_4 = 4\pi, \; b_5 = 6\pi, \; b_6 = 8 $$

$$ \phi_1 = -\frac{24}{(2h+4)} \int_0^{\frac{\pi}{4}} \frac{1}{(\cos\theta)^{2h+4}}\, d\theta $$

and

$$\phi_2 = -8a^3\left\{\int_{u=a^2}^{2a^2} u^b \int_0^{\cos^{-1}\frac{a}{\sqrt{u}}}\left[u - \frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}} d\theta\right.$$

$$\left. + \int_{u=2a^2}^{3a^2} u^b \int_{\cos^{-1}\frac{a}{\sqrt{u-a^2}}}^{\frac{\pi}{4}}\left[u - \frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}} d\theta\right\}$$

$$= -4a^2\left\{\int_{z=a^2}^{2a^2} \frac{1}{z\sqrt{z-a^2}}\left[\int_{u=z}^{z+a^2} u^b (u-z)^{\frac{1}{2}} du\right] dz\right\} \tag{2.10}$$

by using Lemma 1.

The last integral above can be written as follows:

$$\int_{u=z}^{z+a^2} u^b (u-z)^{\frac{1}{2}} du = 2a^{b+\frac{3}{2}} \int_0^{\tan^{-1}\frac{a}{\sqrt{z}}}\left[\frac{1}{(\cos^2\theta)^{b+2}} - \frac{1}{(\cos^2\theta)^{b+1}}\right] d\theta .$$

Now, with the help of Lemmas 4 and 5, ϕ_1 and ϕ_2 can be written explicitly.

3. ONE POINT ON A FACE AND THE OTHER POINT INSIDE

Let $z_2 = 0$. In this case the point P is inside the cube and the point Q is on a face Taking this face as the (x, y)-plane and referring back to the general procedure we have the following: $g_1(v_1)$ remains the same and

$$g_2(v_2) = \frac{v_2^{-\frac{1}{2}}}{2a}, \qquad 0 \le v_2 \le a^2 .$$

Then proceeding as in the general case the density of $u = x^2$, denoted by $f^{(1)}(u)$, is the following:

Theorem 3.

$$f^{(1)}(u) = \frac{1}{a^5}\begin{cases} \rho_1(u), & 0 \le u \le a^2 \\ \rho_2(u), & a^2 \le u \le 2a^2 \\ \rho_3(u), & 2a^2 \le u \le 3a^2 \end{cases}$$

where

$$\rho_1(u) = \pi a^2 u^{\frac{1}{2}} - \pi a u + \frac{2}{3} u^{\frac{3}{2}},$$

$$\rho_1(u) = -\frac{1}{3}a^3 + (a+\pi a)u - 2a^2(u-a^2)^{\frac{1}{2}} - 2u(u-a^2)^{\frac{1}{2}} + \frac{2}{3}(u-a^2)^{\frac{3}{2}}$$

$$-2au\left[\sin^{-1}\frac{a}{\sqrt{u}} - \cos^{-1}\frac{a}{\sqrt{u}}\right] - 4a^2\int_0^{\cos^{-1}\frac{a}{\sqrt{u}}}\left[u - \frac{a^2}{\cos^2\theta}\right]d\theta$$

and

$$\rho_3(u) = -\frac{5}{3}a^3 - au + 2a^2(u-2a^2)^{\frac{1}{2}} + u(u-2a^2)^{\frac{1}{2}}$$

$$= -\frac{1}{3}(u-2a^2)^{\frac{3}{2}} + 2a^3\left[\sin^{-1}\frac{a}{\sqrt{u-a^2}} - \cos^{-1}\frac{a}{\sqrt{u-a^2}}\right]$$

$$-2a(u-a^2)\left[\sin^{-1}\frac{a}{\sqrt{u-a^2}} - \cos^{-1}\frac{a}{\sqrt{u-a^2}}\right]$$

$$-4a^2\int_{\cos^{-1}\frac{a}{\sqrt{u-a^2}}}^{\frac{\pi}{4}}\left[u - \frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}}d\theta.$$

The general h-th moment, for an arbitrary h, can be computed by integrating with the help of Lemmas 1 to 5. The final result is the following, the details of the steps are omitted for saving space:

Theorem 4. For $\Re(h) > -1$,

$$\mathrm{E}(u^h) = a^{2h}\left\{\frac{2\pi}{(2h+3)} + \frac{4}{3(2h+5)} + \frac{1}{(h+1)}\left[\frac{1}{3} + \frac{2^{h+3}}{3} - 5(3^h)\right]\right.$$

$$+\frac{1}{(h+2)}[-1-\pi+2^{h+3}-3^{h+2}]$$

$$-\frac{4}{3}\,{}_2F_1\left(-h,\frac{3}{2};\frac{5}{2};-1\right) - \frac{4}{3}\,{}_2F_1\left(-h,-1,\frac{3}{2};\frac{5}{2};-1\right)$$

$$+\frac{4}{15}\,{}_2F_1\left(-h,\frac{5}{2};\frac{7}{2};-1\right) - \frac{2\sqrt{2}}{(h+2)}\,{}_2F_1\left(h+\frac{5}{2},\frac{1}{2};\frac{3}{2};\frac{1}{2}\right)$$

$$+\frac{2^{h+2}}{3}\,{}_2F_1\left(-h,\frac{3}{2};\frac{5}{2};-\frac{1}{2}\right) + \frac{2^{h+2}}{3}\,{}_2F_1\left(-h,-1,\frac{3}{2};\frac{5}{2};-\frac{1}{2}\right)$$

$$-\frac{2^{h+1}}{15}\,{}_2F_1\left(-h,\frac{5}{2};\frac{7}{2};-\frac{1}{2}\right)$$

$$\left.+\frac{2^{h+3}\sqrt{2}}{(h+2)}F_1\left(\frac{1}{2},h+\frac{5}{2},-h-2;\frac{3}{2};\frac{1}{2},\frac{1}{4}\right) + \psi\right\}$$

where

$$\psi = -4a^2\int_{a^2}^{2a^2} u^b\left\{\int_0^{\cos^{-1}\frac{a}{\sqrt{u}}}\left[u-\frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}}\right\}du$$

$$-4a^2\int_{2a^2}^{3a^2} u^b\left\{\int_{\cos^{-1}\frac{a}{\sqrt{u-a^2}}}^{\frac{\pi}{4}}\left[u-\frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}}\right\}d\theta$$

$$=-2a^3\int_{z=a^2}^{2a^2}\frac{1}{z\sqrt{z-a^2}}\left[\int_{u=z}^{z+a^2} u^b(u-z)^{\frac{1}{2}}\,du\right]dz$$

by using Lemma 1,

$$=-2a^3\int_{z=a^2}^{2a^2}\frac{1}{z\sqrt{z-a^2}}\left[\int_{v=0}^{a^2} v^{\frac{1}{2}}(v-z)^b\,du\right]dz$$

$$-4a^3\int_{z=a^2}^{2a^2} u^b\left[\int_0^{\tan^{-1}\frac{a}{\sqrt{z}}}\left\{\frac{1}{(\cos^2\phi)^{b+2}}-\frac{1}{(\cos^2\phi)^{b+1}}\right\}d\phi\right]dz\,.$$

Since this integral goes into a multiple series for general values of b no further rewriting will be attempted here.

4. THE DISTANCE BETWEEN TWO RANDOM POINTS ON OPPOSITE SIDES OF A CUBE

Let the point P be on a given face. Take one corner of this face as the origin of a rectangular coordinate system and this face on the (x, y) – plane. Let the point Q be on the opposite face. Then the coordinates of P and Q can be written as $P = P(x_1, y_1, 0)$, $Q = Q(x_2, y_2, a)$. The squared distance between P and Q is the given by

$$u = (x_2 - x_1)^2 + (y_2 - y_1)^2 + a^2\,.$$

When a point is uniformly distributed over the surface of a cube of side a the probability that the point is on a given side is $\frac{a^2}{6a^2} = \frac{1}{6}$ and this probability remains the same for all the six faces. Hence, due to symmetry we need to consider only the situation where P is uniformly distributed over the (x, y) – face and Q is independently and uniformly distributed over the opposite face. In other words, we need to assume only that x_j, y_j $j = 1, 2$ are mutually independently and uniformly distributed over $[0, a]$. Note that $u = v_1 + a^2$ where the density of v_1 is $h_2(v_1)$, and from there we have the density of u which is denoted by $f^{(2)}(u)$ in the following theorem.

Theorem 5.

$$f^{(2)}(u)=\frac{1}{a^4}\begin{cases}\psi_1(u), & a^2\le u\le 2a^2\\ \psi_2(u), & 2a^2\le u\le 3a^2\end{cases} \tag{4.1}$$

where

$$\psi_1(u)=\pi a^2-4a(u-a^2)^{\frac{1}{2}}+(u-a^2)$$

and

$$\psi_2(u)=-2a^2-(u-a^2)+4a(u-2a^2)^{\frac{1}{2}}$$
$$+2a^2\left[\sin^{-1}\frac{a}{\sqrt{u-a^2}}-\cos^{-1}\frac{a}{\sqrt{u-a^2}}\right].$$

4.1. *The general h-th moment of u*

Evaluation of the positive integer moments of u is not difficult. But if positive integer moments of $x=u^{1/2}$ are required then one has to consider positive integer and half-integer values of h. For computing the h-th moment of u, for an arbitrary h, one needs the help of the lemmas given earlier. The following is the final result.

Theorem 6. For $\Re(h)>-1$,

$$E(u^h)=a^{2h}\left\{\frac{1}{(h+1)}[-\pi+1-3^{h+1}]+\frac{1}{(h+1)}[2^{h+3}-3^{h+2}-1]\right.$$
$$-\frac{8}{3}\,{}_2F_1\!\left(-h,\frac{3}{2};\frac{5}{2};-1\right)+\frac{2^{h+3}}{3}\,{}_2F_1\!\left(-h,\frac{3}{2};\frac{5}{2};-\frac{1}{2}\right)$$
$$\left.+\sqrt{2}\,\frac{2^{h+\frac{5}{2}}}{(h+1)}F_1\!\left(\frac{1}{2},-h-1,h+\frac{3}{2};\frac{3}{2};\frac{1}{4},\frac{1}{2}\right)\right\}. \tag{4.2}$$

Substituting $h=1/2$, $h=1$ one has the following expressions for the expected values of x and x^2 respectively: From there, the variance of x is readily available

$$E(x)=E(u^{\frac{1}{2}})=a\left\{-\frac{2}{3}\pi+\frac{\sqrt{2}}{5}-\frac{8}{5}\sqrt{3}+\frac{4}{15}-4\ln\sqrt{2}-\ln(\sqrt{2}-1)\right.$$
$$\left.+4\ln(\sqrt{3}-1)+\frac{16}{3}F_1\!\left(\frac{1}{2},-\frac{3}{2},2;\frac{3}{2};\frac{1}{4},\frac{1}{2}\right)\right\} \tag{4.3}$$

and

$$E(x^2)=E(u)=\frac{4}{3}a^2. \tag{4.4}$$

4.2. *Distance between two random points in a square*

Observe that v_1 is the squared distance between two independently and uniformly distributed random points in a square. Arbitrary moments of this squared distance in a square is then $E(v_1^h)$ for an arbitrary h, which is available by direct integration. The result is the following:

Corollary 4.1. For $\Re(h) > -1$

$$E(v_1^h) = a^{2h}\left\{2(1-2^{h+1})\left[\frac{1}{(h+1)}+\frac{1}{(h+2)}\right]-\frac{8}{(2h+3)}\right.$$

$$\left.+\frac{8}{3}\,{}_2F_1\left(-h,\frac{3}{2};\frac{5}{2};-1\right)+\frac{2\sqrt{2}}{(h+1)}\,{}_2F_1\left(h+\frac{3}{2},\frac{1}{2};\frac{3}{2};\frac{1}{2}\right)\right\}. \qquad (4.5)$$

5. RANDOM POINTS ON ADJACENT FACES

Put $y_2 = a$, $z_1 = 0$. Then the point P is on the (x, y) - plane and the other point Q is on the face $y_2 = a$. Since y_j and $a - y_j$ are identically distributed, and the same goes for x_j as well as z_j, this case is the same as saying that the two points are on adjacent faces. The distance between P and Q in this case is also the length of the secant across two adjacent faces or random path across two adjacent faces when the points are independently and uniformly distributed over the respective faces.

Referring back to our notations we have

$$f_1(u_1) = \frac{1}{a^2}\left\{au_1^{-\frac{1}{2}}-1\right\}, \qquad 0 \le u_1 \le a^2$$

$$f_2(u_2) = \frac{u_2^{-\frac{1}{2}}}{2a}, \qquad 0 \le u_2 \le a^2$$

and let

$$f_3(u_3) = \frac{u_3^{-\frac{1}{2}}}{2a}, \qquad 0 \le u_3 \le a^2 .$$

Then proceeding as before the various expressions are the following:

$$g_1(v_1) = \frac{1}{a^3}\begin{cases}\dfrac{\pi a}{2} - v_1^{\frac{1}{2}}, 0 \le v_1 \le a^2 \\ -a + (v_1 - a^2)^{\frac{1}{2}} + a\left[\sin^{-1}\dfrac{a}{\sqrt{v_1}} - \cos^{-1}\dfrac{a}{\sqrt{v_1}}\right], \\ a^2 \le v_1 \le 2a^2\end{cases}$$

and let

$$g_2(v_2) = f_3(v_2)\ .$$

Consequently, the density of $u = x^2$, where x is the distance between P and Q, denoted by $f^{(3)}(u)$, is the following:

Theorem 7.

$$f^{(3)}(u) = \frac{1}{a^4}\begin{cases} \eta_1(u), & 0 \le u \le a^2 \\ \eta_2(u), & a^2 \le u \le 2a^2 \\ \eta_3(u), & 2a^2 \le u \le 3a^2 \end{cases} \tag{5.1}$$

where

$$\eta_1(u) = \frac{\pi a}{2} u^{\frac{1}{2}} - \frac{\pi}{4} u\,,$$

$$\eta_2(u) = \frac{\pi a^2}{4} + \frac{\pi}{4} u - a(u - a^2)^{\frac{1}{2}} - \frac{u}{2}\left[\sin^{-1}\frac{a}{\sqrt{u}} - \cos^{-1}\frac{a}{\sqrt{u}}\right]$$

$$- 2a\int_0^{\cos^{-1}\frac{a}{\sqrt{u}}}\left[u - \frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}} d\theta$$

and

$$\eta_3(u) = -a^2 + a(u - 2a^2)^{\frac{1}{2}}$$

$$+ \frac{1}{2}(u - a^2)\left[\sin^{-1}\frac{a}{\sqrt{u - a^2}} - \cos^{-1}\frac{a}{\sqrt{u - a^2}}\right]$$

$$+ a^2\left[\sin^{-1}\frac{a}{\sqrt{u - a^2}} - \cos^{-1}\frac{a}{\sqrt{u - a^2}}\right]$$

$$- 2a\int_{\cos^{-1}\frac{a}{\sqrt{u-a^2}}}^{\frac{\pi}{4}}\left[u - \frac{a^2}{\cos^2\theta}\right]^{\frac{1}{2}} d\theta\,.$$

In order to see that $f^{(3)}(u)$ is in fact a density we need to evaluate the integrals in $\eta_2(u)$ and $\eta_3(u)$. The steps in this procedure are also needed when computing the arbitrary moments u in this case. This is done with the help of Lemma 1 and the final result is the following:

Theorem 8. For $\Re(h) > -1$

$$
\begin{aligned}
E(u^h) = a^{2h}\Big\{ & \frac{1}{(h+1)}\Big[2^{h+1} - 3^{h+1} - \frac{\pi}{4}\Big] - \frac{\pi}{4(h+2)} + \frac{\pi}{(2h+3)} \\
& + \frac{2^{h+1}}{3}\, {}_2F_1\Big(-h, \frac{3}{2}; \frac{5}{2}; -\frac{1}{2}\Big) - \frac{2}{3}\, {}_2F_1\Big(-h, \frac{3}{2}; \frac{5}{2}; -1\Big) \\
& - \frac{1}{(h+2)} \frac{1}{\sqrt{2}}\, {}_2F_1\Big(-h + \frac{5}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{2}\Big) \\
& + \frac{2^{h+\frac{3}{2}}}{(h+1)} F_1\Big(\frac{1}{2}, -h-1, h+\frac{3}{2}; \frac{3}{2}; \frac{1}{4}, \frac{1}{2}\Big) \\
& + \frac{2^{h+\frac{3}{2}}}{(h+2)} F_1\Big(\frac{1}{2}, -h-2, h+\frac{5}{2}; \frac{3}{2}; \frac{1}{4}, \frac{1}{2}\Big) \\
& - \frac{2^{h+\frac{1}{2}}}{(h+1)} F_1\Big(\frac{1}{2}, -h-1, h+\frac{3}{2}; \frac{3}{2}; \frac{3}{2}; \frac{1}{4}, \frac{1}{2}\Big) + \phi \Big\}
\end{aligned} \tag{5.2}
$$

where ϕ is coming from the last two integrals in $f^{(3)}(u)$, which is given below.

$$
\phi = -a^2\left\{\int_{z=a^2}^{2a^2} \frac{1}{z\sqrt{z-a}}\left[\int_{u=z}^{z+a^2} u^h (u-z)^{\frac{1}{2}}\, du\right] dz\right\}. \tag{5.3}
$$

Put $v = u - z$, $z = a^2 t$, $v = a^2 w$ to rewrite

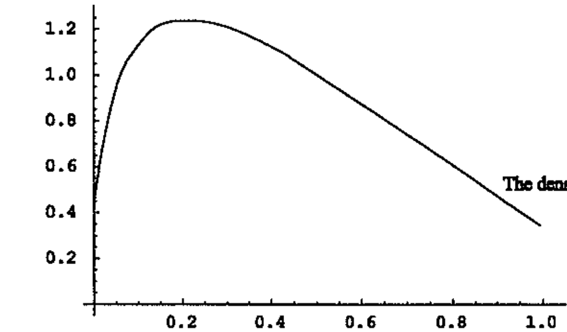
$$
\phi = -a^{2h+4}\left\{\int_0^1 w^{\frac{1}{2}}\left[\int_0^1 \frac{(t+(1+w))^h}{\sqrt{t}(1+t)}\, dt\right] dw\right\}. \tag{5.4}
$$

For positive integer values of h this double integral is easily evaluated either by expanding $(t + (1 + w))^h$ with the help of a binomial expansion and then integrating out or by starting with (5.3), writing the inner integral as $\int_0^{a^2} v^{1/2}(v + z)^h dv$, expanding $(v + z)^h$ and then integrating out. But for arbitrary h it does not seem that ϕ can be put into any simple form. One can write ϕ in terms of a general triple hypergeometric series. This will be given here in order to save space.

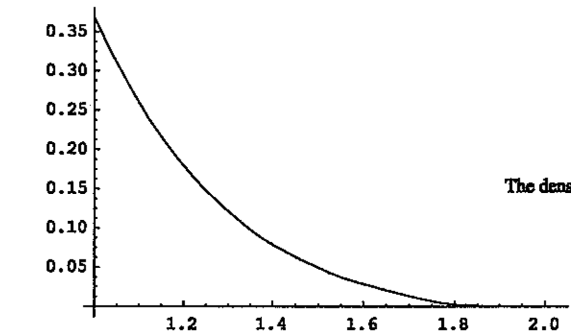
6. GRAPHS, SIMULATION RESULTS AND REMARKS

The following are the graphs of the densities, drawn by using a mathematica program, showing the behaviour of the tail and mid-regions along with the combined graphs for a cube of side 1, that is $a = 1$.

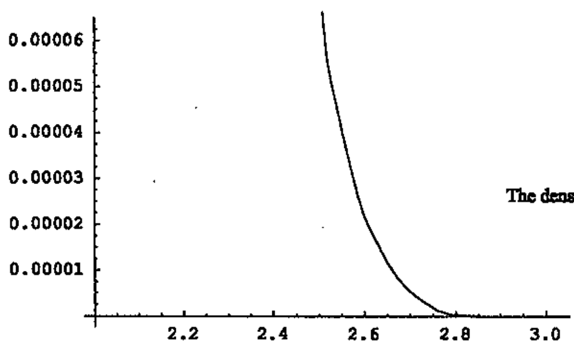
The simplest case is the density in Theorem 5 which behaves like an exponential curve. The next ones correspond to the densities in Theorem 7, Theorem 3 and Theorem 1 respectively. The curves seem to approximate progres-



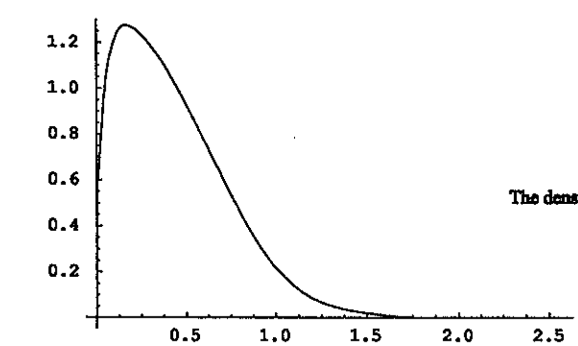
The density in Theorem 1 for α=1, left tail



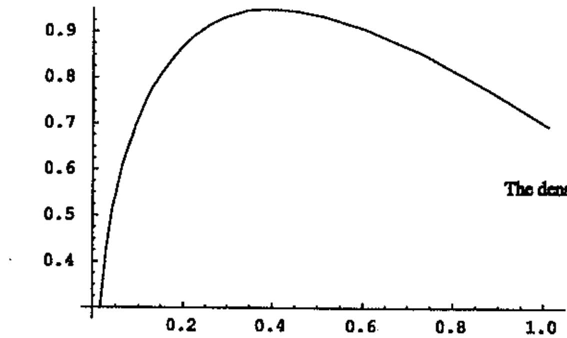
The density in Theorem 1 for α=1, middle portion



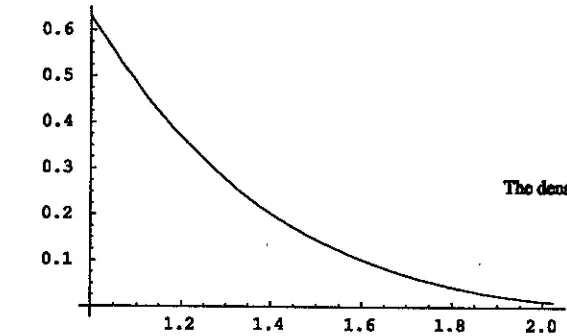
The density in Theorem 1 for α=1, right tail



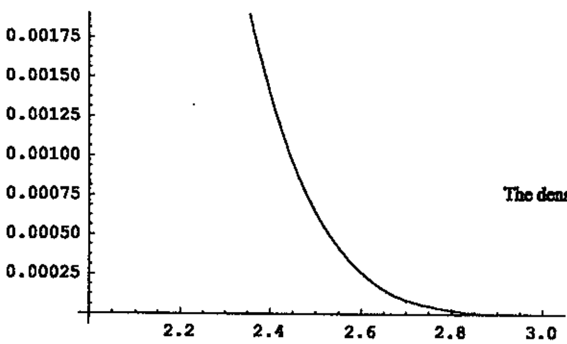
The density in Theorem 1 for α=1



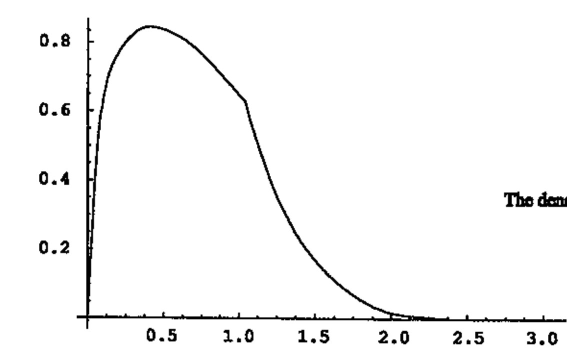
The density in Theorem 3 for $\alpha=1$, left tail



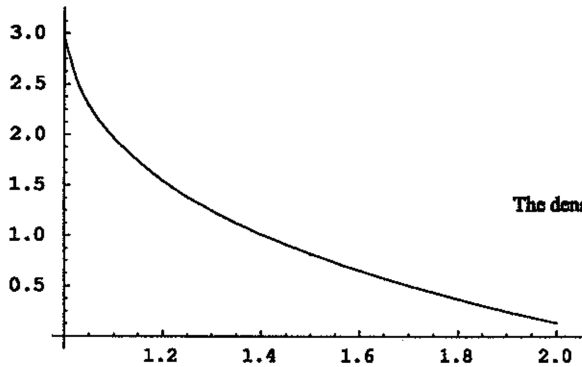
The density in Theorem 3 for $\alpha=1$, middle portion



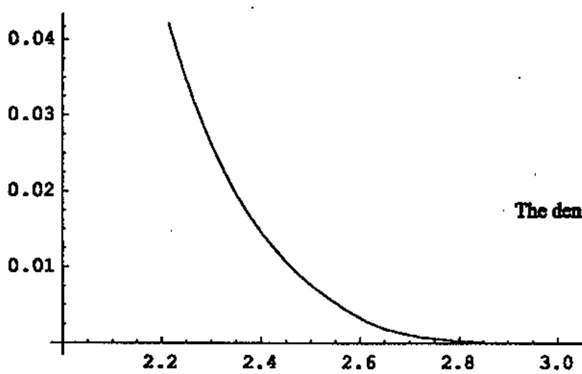
The density in Theorem 3 for $\alpha=1$, right tail



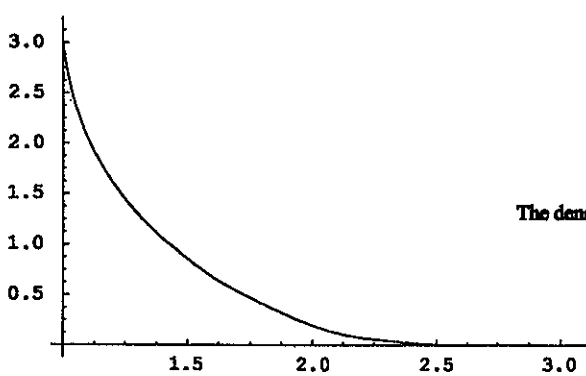
The density in Theorem 3 for $\alpha=1$



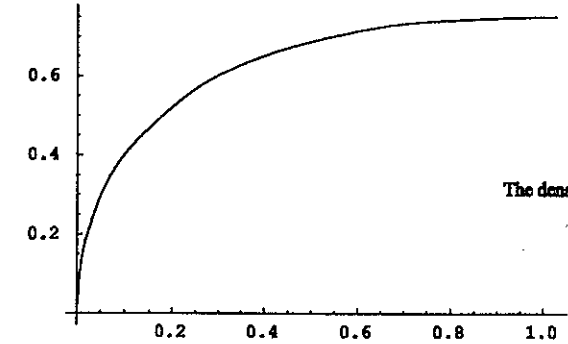
The density in Theorem 5 for α=1, left tail



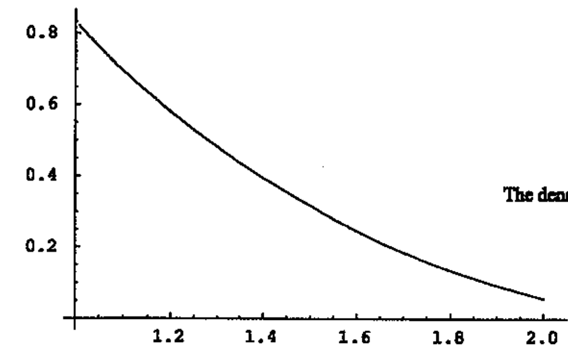
The density in Theorem 5 for α=1, right tail



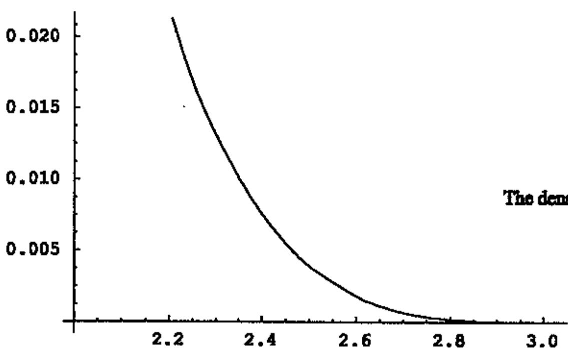
The density in Theorem 5 for α=1



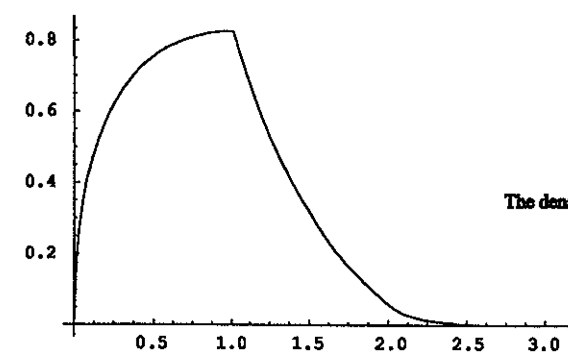
The density in Theorem 7 for $\alpha=1$, left tail



The density in Theorem 7 for $\alpha=1$, middle portion



The density in Theorem 7 for $\alpha=1$, right tail



The density in Theorem 7 for $\alpha=1$

sively to that of a gamma type density. The right tail contributes less and less as one goes from Theorem 7 to Theorem 3 to Theorem 1. Fairly good approximations to the densities are available by deleting fully the right tail terms in Theorem 3 and 1 and the terms containing the integrals in Theorem 7.

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RIASSUNTO

Distanze tra punti aleatori di un cubo

Vengono fornite le forme esplicite delle densità e dei momenti relativi alla distanza tra due punti aleatori distribuiti in modo uniforme ed indipendente all'interno di un cubo. Casi particolari riguardano situazioni dove i punti si trovano su facce opposte oppure adiacenti. Espressioni esatte per momenti arbitrari sono ottenute in termini di funzioni di Gauss, Appell e Lauricella.

SUMMARY

Distance between random points in a cube

The explicit forms of the exact density and the exact arbitrary moments of the distance between two random points when the points are independently and uniformly distributed inside a cube are given. Special cases considered include the density and arbitrary moments of the distance between two independent random points where one is inside the cube and the other is on a face, points on adjacent feces, points on opposite faces and random points inside a square. Exact arbitrary moments are obtained in terms of Gauss' hypergeometric functions and Appell's functions or Lauricella functions.

BIOMETRICS

JOURNAL OF THE BIOMETRIC SOCIETY

Vol. 54, No. 4 December 1998

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