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POLLUTION BY VEHICULAR TRAVELS FROM SATELLITE TOWNSHIPS TO THE CITY

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SUMMARY

Pollutants emitted by vehicles, noise pollution and other adverse effects on the environment can be measured in terms of the expected distance travelled. Various travel plans of vehicles going from a suburb to the city core or vice versa when each region is circular in nature and within these regions the starting and destination points are uniformly distributed have already been examined. In the present article the assumption of uniformity is relaxed and some general families of distributions are associated with the random points. Various travel plans and the expected distances are computed. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS distance between random points; travel distance; pollution problem; circular cities; rectangular network

1. INTRODUCTION

Smeed (1961), Fairthorne (1964, 1965), Einhorn (1967), Holroyd (1969) and Pearce (1974) examine various travel plans inside a circular city, polar versus rectangular road networks in a circular city, ring roads and related aspects. Travel distance in an expanding city is considered in Tan (1966). Traffic flow on a divided highway is examined in Rényi (1964). Low-density traffic is studied in Weiss and Herman (1962).

Pollutants emitted by vehicles, noise pollution, road surface wear and tear and other environmental hazards associated with vehicular travel can be measured in terms of expected travel distances. Mathai (1998) examined vehicle travel from a suburb to the city core or vice versa when the suburb and the city core are assumed to be circular in nature with polar or rectangular road networks. Expected travel distances under various travel plans are examined when the starting and ending points are assumed to be independently and uniformly distributed within each circular region. If there is traffic congestion near the city centre or near the exit point or bad road conditions away from the city centre, then such factors force us to abandon the assumption of uniform random points. Here we consider the situation of some general classes of distributions

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associated with the random points and examine the expected travel distances under various travel plans. We will also examine the relative merits of the various travel plans. Air pollution and acid rain problems are studied in Provost and Barnwal (1993) and Provost and Cheong (1998) by looking at properties of random points in Gaussian random fields. Total amount of pollutants emitted to the atmosphere by a vehicular travel will be taken as a constant multiple of the number of vehicles and the expected travel distance for a typical travel.

2. VARIOUS TRAVEL PLANS

Consider the ith satellite township for $i = 1, \ldots, k$. Let C_i and C^* be circles of radii r_i and r^* and centres O_i and O^* representing the ith township and the city core respectively. Consider the line joining the centres O_i and O^*. Let the major exit point from C_i and the entry point to C^* be on this line as shown in Figure 1. If they are not both on the line joining the centres the problem can still be studied with minor modifications. Taking into consideration all travels originating from inside C_i and ending in all possible points in C^* we may assume the starting point P_i and the ending point P^* for a typical trip to be random points. Thus the expected travel distance between P_i and P^* along a given path, when P_i and P^* are distributed inside C_i and C^* according to some specific probability laws, is an indicator of the amount of pollutants emitted by the vehicle or other environmental hazards. If no other factors are involved, then the assumption that P_i and P^* are independently and uniformly distributed inside C_i and C^* is a reasonable one. But if there is traffic congestion near the centres O_i and O^*, a natural situation, or if the road system is poorer as one moves away from the centres, then uniform distributions are not reasonable models for P_i and P^*. We will start a general type-1 beta family of distributions for ρ_i, the distance of P_i from O_i, and a similar model for ρ_i^*, the distance of P^* from O^* when C_i is under consideration. For the time being, let us assume that ρ_i and θ_i, the angle O_iP_i makes with the x-axis, which is taken as the line passing through the centres O_i and O^*, are independently distributed, where θ_i is uniform over $[0, 2\pi]$ and ρ_i has the density

$$g_i(\rho_i)\mathrm{d}\rho_i = \frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i)} \frac{\rho_i^{\alpha_i - 1}(r_i - \rho_i)^{\beta_i - 1}}{r_i^{\alpha_i + \beta_i - 1}} \mathrm{d}\rho_i, \quad 0 \leqslant \rho_i \leqslant r_i, \quad \alpha_i > 0, \quad \beta_i > 0$$

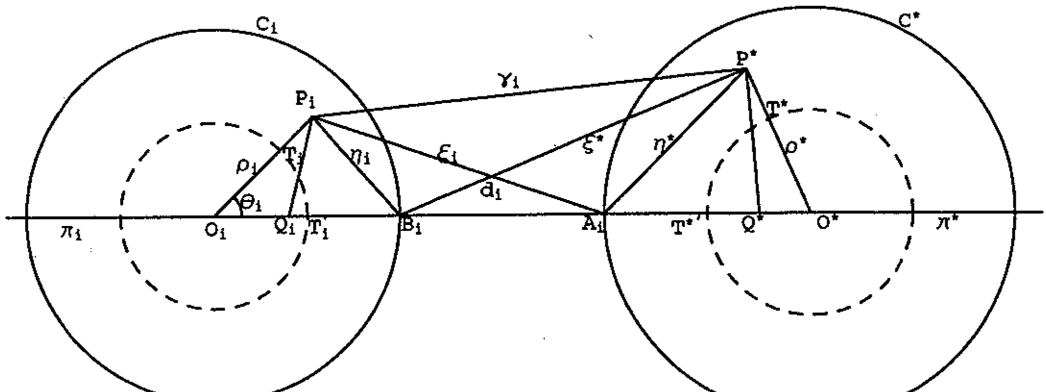

Figure 1. i-th circular township and the circular city

and zero elsewhere, so that the joint density of ρ_i and θ_i is the product of $g_i(\rho_i)$ and the uniform density over $[0, 2\pi]$. Let us denote the travel distance under travel plan j associated with C_i by t_{ij} and let d_i be the distance between the exit point B_i from C_i and the entry point A_i to C^*. Consider the following travel plans:

Plan 1: Going from P_i to O_i then to B_i to A_i.

$$t_{i1} = \rho_i + r_i + d_i$$

If the road system in C_i is of polar type, then the natural travel plan will be to come to the centre and exit through the major exit road. For the time being, we consider only one exit point, B_i. This B_i could be a fixed point on the circumference of C_i or a random point independently and uniformly distributed over the circumference. The final results will remain the same. If the road network inside C_i is of rectangular type, then the total distance from P_i to B_i along such a system is equivalent to coming from P_i to Q_i, the foot of the perpendicular from P_i to O_iO^* and then to B_i to A_i. Thus we have

Plan 2: Going from P_i to Q_i and then to B_i to A_i.

$$t_{i2} = \rho_i|\sin\theta_i| + (r_i - \rho_i\cos\theta_i) + d_i$$

If the network inside C_i is very good with all sorts of cross roads, then we can consider two other travel plans of going from P_i to B_i directly and then to A_i and going from P_i directly to A_i.

Plan 3: Going from P_i to B_i directly and then to A_i.

$$\begin{aligned} t_{i3} &= \eta_i + d_i \\ &= [\rho_i^2 + r_i^2 - 2r_i\rho_i\cos\theta_i]^{\frac{1}{2}} + d_i. \end{aligned}$$

Plan 4: Going from P_i directly to A_i.

$$t_{i4} = \zeta_i = [\rho_i^2 + (r_i + d_i)^2 - 2\rho_i(r_i + d_i)\cos\theta_i]^{\frac{1}{2}}.$$

Let the corresponding plans in C^* be denoted by Plans 1*, 2*, 3*, 4*, respectively. When the road system is of polar type it is worth investigating whether it is advantageous to put a ring road at a fixed distance a_i from O_i so that the travel plan will be to travel from P_i to the ring road and then along the circumference of this ring road to the line O_iO^* then to B_i. Then the travel plan and the corresponding distance are the following:

Plan 5: Going from P_i to the ring road, along the ring road to O_iO^* and then to B_i.

$$t_{i5} = |\rho_i - a_i| + a_i\theta_i + r_i - a_i$$

where θ_i is measured in radians. A sixth plan that we can consider is going from P_i directly to P^*. This is feasible if there are many exit and entry points, many connecting roads, etc.

Plan 6: Going from P_i directly to P^*.

$$t_{i6} = \gamma_i.$$

As noted in Mathai (1999), no convenient expression for the expected distance under Plan 6 is available, other than an expression in terms of Lauricella function, even when the points P_i and P^* are uniformly distributed. Under the type-1 beta model also one can arrive at a similar expression, but this will be quite involved. Hence we will consider only Plans 1–5 and the corresponding plans 1*–5*.

If the random part of the travel distance associated with C_i is denoted by u_i, the corresponding random part associated with C^* by v_i, and the constant part of the travel distance when the ith township is under consideration by f_i, then for a particular travel plan the total distance from a random point inside C_i to a random point inside C^* can be denoted by

$$z_i = u_i + v_i + f_i, \quad i = 1, \ldots, k.$$

Then the grand total distance of typical travels from all the k townships to the city core, under a given travel plan, is given by

$$z = \sum_{j=1}^{k} z_j = \sum_{j=1}^{k} u_j + \sum_{j=1}^{k} v_j + \sum_{j=1}^{k} f_j$$

where, without loss of generality, $v_1, \ldots, v_k$ are independently and identically distributed and $u_1, \ldots, u_k$ are independently distributed. Then the expected travel distance is available by taking the expected value of z above.

3. EXPECTED TRAVEL DISTANCES IN C_i UNDER VARIOUS PLANS

From Plan 1 and the type-1 beta model we have

$$\begin{aligned} E(t_{i1}) &= r_i + d_i + E(\rho_i) \\ &= r_i d_i + \frac{1}{2\pi} \frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i)} \frac{\int_0^{2\pi} \int_0^{r_i} \rho_i^{\alpha_i - 1} (r_i - \rho_i)^{\beta_i - 1} \mathrm{d}\rho_i \mathrm{d}\theta_i}{r_i^{\alpha_i + \beta_i - 1}} \\ &= r_i + d_i + \frac{\alpha_i}{\alpha_i + \beta_i} r_i. \end{aligned}$$

From Plan 2 and the type-1 beta model we have

$$\begin{aligned} E(t_{i2}) &= r_i + d_i + \frac{1}{2\pi} \int_0^{2\pi} [|\sin \theta_i| - \cos \theta_i] \mathrm{d}\theta_i \\ &\quad \times \frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i)} \frac{\int_0^{r_i} \rho_i^{1+\alpha_i - 1} (r_i - \rho_i)^{\beta_i - 1} \mathrm{d}\rho_i}{r_i^{\alpha_i + \beta_i - 1}} \\ &= r_i + d_i + \frac{2}{\pi} \frac{\alpha_i}{\alpha_i + \beta_i} r_i. \end{aligned}$$

Is the rectangular network of roads better than the polar system of roads? We can examine by comparing $E(t_{i1})$ and $E(t_{i2})$. Note that since $\frac{2}{\pi} < 1$ for all admissible values of α_i and β_i, the rectangular system is better than the polar system with respect to the expected travel distance. From Plan 3 and the type-1 beta model one can compute $E(t_{i3})$ by using the following technique.

$$E(t_{i3}) = d_i + E[\rho_i^2 + r_i^2 - 2r_i\rho_i\cos\theta_i]^{\frac{1}{2}}$$
$$= d_i + \frac{1}{2\pi}\frac{\Gamma(\alpha_i+\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i)}\int_0^{2\pi}\int_0^{r_i}(\rho_i^2 + r_i^2 - 2r_i\rho_i\cos\theta_i)^{\frac{1}{2}}$$
$$\times\frac{\rho_i^{\alpha_i-1}(r_i-\rho_i)^{\beta_i-1}\mathrm{d}\rho_i\mathrm{d}\theta_i}{r_i^{\alpha_i+\beta_i-1}}.$$

Put $\rho_i = r_i u$. Then

$$E(t_{i3}) = d_i + \frac{r_i}{2\pi}\frac{\Gamma(\alpha_i+\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i)}\int_{u=0}^{1}u^{\alpha_i-1}(1-u)^{\beta_i-1}$$
$$\times\left\{\int_0^{2\pi}[1+u^2-2u\cos\theta_i]^{\frac{1}{2}}\mathrm{d}\theta_i\right\}\mathrm{d}u.$$

But note that

$$(1+u^2-2u\cos\theta_j)^{\frac{1}{2}} = (1-u\mathrm{e}^{i\theta_j})^{\frac{1}{2}}(1-u\mathrm{e}^{-i\theta_j})^{\frac{1}{2}},\quad i=\sqrt{-1}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_m}{m!}\frac{\left(-\frac{1}{2}\right)_n}{n!}u^{m+n}\mathrm{e}^{i\theta_j(m-n)}$$

by expanding each factor with the help of a binomial expansion, where, for example.

$$(a)_m = a(a+1)\ldots(a+m-1),\quad (a)_0 = 1,\quad a\neq 0.$$

Let $m+n=p$. Then the above double sum reduces to

$$\sum_{p=0}^{\infty}\sum_{m=0}^{p}\frac{\left(-\frac{1}{2}\right)_m}{m!}\frac{\left(-\frac{1}{2}\right)_{p-m}}{(p-m)!}u^p\mathrm{e}^{i\theta_j(2m-p)}.$$

Integrating out θ_j we note that all terms are zeros, except when $p=2m$ and the corresponding integral yields 2π. Then putting $p=2m$ we have

$$E(t_{i3}) = d_i + \frac{\Gamma(\alpha_i+\beta_i)r_i}{\Gamma(\alpha_i)\Gamma(\beta_i)}\sum_{m=0}^{\infty}\left[\frac{\left(-\frac{1}{2}\right)_m}{m!}\right]^2\int_0^1 u^{2m+\alpha_i-1}(1-u)^{\beta_i-1}\mathrm{d}u$$
$$= d_i + r_i\sum_{m=0}^{\infty}\frac{\alpha_i(\alpha_i+1)\ldots(\alpha_i+2m-1)}{(\alpha_i+\beta_i)\ldots(\alpha_i+\beta_i+2m-1)}\frac{\left(-\frac{1}{2}\right)_m}{m!}\frac{\left(-\frac{1}{2}\right)_m}{m!}.$$

Converting in terms of the Pochhammer symbol and then writing the resulting expression as a hypergeometric function we have

$$E(t_{i3}) = d_i + \frac{\alpha_i r_i}{\alpha_i + \beta_i}\,{}_3F_2\left(\frac{\alpha_i + 1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i + \beta_i + 1}{2}, 1; 1\right).$$

A few values of the expected distance $E(t_{i3})$ for $i = 1$, $r_1 = 1$, $d_1 = 1$ are given in Table I.

From Plan 4 and the type-1 beta model we have

$$E(t_{i4}) = E[\rho_i^2 + (r_i + d_i)^2 - 2\rho_i(r_i + d_i)\cos\theta_i]^{\frac{1}{2}}.$$

Let $r_i + d_i = R_i$. Put $\rho_i = R_i u$. Then

$$E(t_{i4}) = \frac{1}{2\pi}\frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i) r_i^{\alpha_i+\beta_i-1}} \times \int_0^{2\pi}\int_0^{r_i/R_i} \rho_i^{\alpha_i-1}(r_i - \rho_i)^{\beta_i-1} R_i^2[1 + u^2 - 2u\cos\theta_i]^{\frac{1}{2}}\mathrm{d}\theta_i\mathrm{d}u.$$

Table I. $E(t_{i3})$ for $i = 1$, $r_1 = 1$, $d_1 = 1$ and for various values of $d_1 = d$, $\alpha_1 = \alpha$ and $\beta_1 = \beta$

	$\beta = 0{\cdot}5$	$\beta = 0{\cdot}6$	$\beta = 0{\cdot}7$	$\beta = 0{\cdot}8$
$\alpha = 0{\cdot}5$	1·09953	1·08582	1·07482	1·06585
$\alpha = 0{\cdot}6$	1·11037	1·09619	1·08463	1·07506
$\alpha = 0{\cdot}7$	1·11979	1·10535	1·09342	1·08343
$\alpha = 0{\cdot}8$	1·12806	1·11352	1·10135	1·09107
	$\beta = 0{\cdot}5$	$\beta = 0{\cdot}6$	$\beta = 0{\cdot}7$	$\beta = 0{\cdot}8$
$\alpha = 1$	1·14197	1·12750	1·11516	1·10455
$\alpha = 2$	1·18333	1·17081	1·15955	1·14937
$\alpha = 3$	1·20431	1·19374	1·18398	1·17495
	$\beta = 1$	$\beta = 2$	$\beta = 3$	
$\alpha = 0{\cdot}5$	1·05220	1·02198	1·01211	
$\alpha = 0{\cdot}6$	1·06028	1·02633	1·01475	
$\alpha = 0{\cdot}7$	1·06778	1·03060	1·01742	
$\alpha = 0{\cdot}8$	1·07475	1·03479	1·02011	
	$\beta = 1$	$\beta = 2$	$\beta = 3$	
$\alpha = 1$	1·08734	1·04290	1·02551	
$\alpha = 2$	1·13177	1·07770	1·05129	
$\alpha = 3$	1·15881	1·10411	1·07361	

Now, going through the same steps as in the case of t_{i3} we see that

$$E(t_{i4}) = R_i^2 \frac{\Gamma(\alpha_i+\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i)} \frac{1}{r_i^{\alpha_i+\beta_i-1}} \sum_{m=0}^{\infty} \left[\frac{(-\frac{1}{2})_m}{m!}\right]^2 \int_0^{r_i/R_i} (R_i u)^{\alpha_i-1} u^{2m} (r_i - R_i u)^{\beta_i-1} \mathrm{d}u$$

$$= R_i \frac{\alpha_i}{\alpha_i+\beta_i} \sum_{m=0}^{\infty} \left[\frac{(-\frac{1}{2})_m}{m!}\right]^2 \frac{(\alpha_i+1)\ldots(\alpha_i+2m-1)}{(\alpha_i+\beta_i+1)\ldots(\alpha_i+\beta_i+2m-1)} (r_i/R_i)^{2m}$$

$$= R_i \frac{\alpha_i}{\alpha_i+\beta_i} {}_3F_2\left(\frac{\alpha_i+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i+\beta_i+1}{2}, 1; \frac{r_i}{R_i}\right).$$

The series is evidently convergent, since $\frac{r_i}{R_i} < 1$ for $d_i > 0$. When $d_i = 0$ it reduces to the case of $E(t_{i3})$. Table II gives $E(t_{i4})$ for $i = 1$ and for various values of the parameters d_i, α_i, β_i.

From Plan 5 and the type-1 beta model we have

$$E(t_{i5}) = r_i - a_i - a_i E(\theta_i) + E|\rho_i - \alpha_i|.$$

Note that

$$E(\theta_i) = \frac{1}{2\pi} \int_0^{2\pi} \theta_i \mathrm{d}\theta_i = \pi.$$

Table II. $E(t_{i4})$ for $i = 1$, $r_1 = 1$ and for various values of $d_1 = d$, $\alpha_1 = \alpha$, $\beta_1 = \beta$

	$\beta = 0{\cdot}5$	$\beta = 1{\cdot}0$	$\beta = 1{\cdot}5$	$\beta = 2{\cdot}0$	$\beta = 2{\cdot}5$
$d = 1$					
$\alpha = 0{\cdot}5$	2·0474	2·0675	2·0791	2·0869	2·0924
$\alpha = 1{\cdot}0$	2·0252	2·0421	2·0541	2·0632	2·0702
$\alpha = 1{\cdot}5$	2·0157	2·0288	2·0394	2·0481	2·0552
$\alpha = 2{\cdot}0$	2·0108	2·0210	2·0300	2·0378	2·0446
$\alpha = 2{\cdot}5$	2·0079	2·0160	2·0236	2·0305	2·0367
$d = 2$					
$\alpha = 0{\cdot}5$	3·0314	3·0447	3·0524	3·0575	3·0611
$\alpha = 1{\cdot}0$	3·0167	3·0279	3·0359	3·0419	3·0465
$\alpha = 1{\cdot}5$	3·0105	3·0191	3·0261	3·0319	3·0366
$\alpha = 2{\cdot}0$	3·0072	3·0139	3·0199	3·0251	3·0296
$\alpha = 2{\cdot}5$	3·0052	3·0106	3·0157	3·0203	3·0244
$d = 3$					
$\alpha = 0{\cdot}5$	4·0235	4·0334	4·0392	4·0430	4·0457
$\alpha = 1{\cdot}0$	4·0125	4·0209	4·0269	4·0313	4·0348
$\alpha = 1{\cdot}5$	4·0078	4·0143	4·0196	4·0239	4·0274
$\alpha = 2{\cdot}0$	4·0054	4·0104	4·0149	4·0188	4·0221
$\alpha = 2{\cdot}5$	4·0039	4·0080	4·0117	4·0152	4·0183
$d = 4$					
$\alpha = 0{\cdot}5$	5·0188	5·0267	5·0313	5·0344	5·0365
$\alpha = 1{\cdot}0$	5·0100	5·0167	5·0215	5·0250	5·0278
$\alpha = 1{\cdot}5$	5·0063	5·0114	5·0157	5·0191	5·0219
$\alpha = 2{\cdot}0$	5·0043	5·0083	5·0119	5·0150	5·0177
$\alpha = 2{\cdot}5$	5·0031	5·0064	5·0094	5·0121	5·0146

Then

$$E(t_{i5}) = r_i + (\pi - 1)a_i + E|\rho_i - a_i|$$

Let us evaluate $E|\rho_i - a_i|$.

$$E|\rho_i - a_i| = \frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i)r_i^{\alpha_i+\beta_i-1}}\left\{\int_0^{a_i}(a_i - \rho_i)\rho_i^{\alpha_i-1}(r_i - \rho_i)^{\beta_i-1}\mathrm{d}\rho_i\right.$$

$$\left. + \int_{a_i}^{r_i}(\rho_i - \alpha_i)\rho_i^{\alpha_i-1}(r_i - \rho_i)^{\beta_i-1}\mathrm{d}\rho_i)\right\}$$

$$= -a_i + \frac{\alpha_i}{\alpha_i + \beta_i}r_i + 2a_i I(\alpha_i, \beta_i; a_i/r_i) + 2r_i\frac{\alpha_i}{\alpha_i + \beta_i}I(\alpha_i + 1, \beta_i; a_i/r_i)$$

where $I(\alpha,\beta;\delta)$ represents the incomplete beta function

$$I(\alpha, \beta; \delta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\delta} x^{\alpha-1}(1 - x)^{\beta-1}\mathrm{d}x.$$

Then

$$E(t_{i5}) = r_i + (\pi - 2)a_i + \frac{\alpha_i}{\alpha_i + \beta_i}r_i$$

$$+ 2a_i I(\alpha_i, \beta_i; a_2/r_i) - 2r_i\frac{\alpha_i}{\alpha_i + \beta_i}I(\alpha_i + 1, \beta_i; a_i/r_i).$$

For example, let $\alpha = 1$. This can be interpreted as the traffic congestion is more and more nearer to the centre O_i or it decreases as one moves away from the centre. Then by evaluating the above expression at $\alpha = 1$ we have

$$E(t_{i5}) = \pi a_i + \frac{r_i\beta_i}{\beta_i + 1} + \frac{2r_i}{\beta_i + 1}\left(1 - \frac{a_i}{r_i}\right)^{\beta_i+1}.$$

Where should the ring road be placed so that $E(t_{i5})$ is a minimum? This can be done by minimizing the above with respect to a_i. The global minimum is attained at

$$a_i = r_i[1 - (\pi/2)^{1/\beta_i}].$$

Since $1 - (\pi/2)^{1/\beta_i} < 0$, this is not an admissible value. The minimum in the range $0 \leqslant a_i \leqslant r_i$ is attained at $a_i = 0$. This means that it is preferable to come through the centre O_i rather than along the ring road in this case. The Table III gives the expected travel distance for various values of a_i, α_i, β_i.

4. COMBINING VARIOUS TRAVEL PLANS

Suppose the travel distance inside the ith satellite township and that inside the city core are independently distributed with ρ_i having type-1 beta density with parameters (α_i, β_i) and ρ_i^* with

Table III. $E(t_{i5})$ for $i = 1$, $r_1 = 1$ and for various values of $a_1 = a$, $\alpha_1 = \alpha$, $\beta_1 = \beta$

	$\beta = 0{\cdot}5$	$\beta = 1{\cdot}0$	$\beta = 1{\cdot}5$	$\beta = 2{\cdot}0$	$\beta = 2{\cdot}5$
$a = 0$					
$\alpha = 0{\cdot}5$	1·500	1·6667	1·7500	1·8000	1·8333
$\alpha = 1{\cdot}0$	1·3333	1·5000	1·6000	1·6667	1·7143
$\alpha = 1{\cdot}5$	1·2500	1·4000	1·5000	1·5714	1·6250
$\alpha = 2{\cdot}0$	1·2000	1·3333	1·4286	1·5000	1·5556
$\alpha = 2{\cdot}5$	1·1667	1·2857	1·3750	1·4444	1·5000
$a = 0{\cdot}2$					
$\alpha = 0{\cdot}5$	1·8059	1·9157	1·9847	2·0304	2·0624
$\alpha = 1{\cdot}0$	1·6809	1·7683	1·8426	1·9003	1·9447
$\alpha = 1{\cdot}5$	1·6271	1·6863	1·7515	1·8092	1·8572
$\alpha = 2{\cdot}0$	1·6001	1·6363	1·6896	1·7427	1·7902
$\alpha = 2{\cdot}5$	1·5856	1·6043	1·6458	1·6927	1·7376
$a = 0{\cdot}4$					
$\alpha = 0{\cdot}5$	2·1813	2·2097	2·2446	2·2746	2·2989
$\alpha = 1{\cdot}0$	2·1273	2·1166	2·1376	2·1660	2·1941
$\alpha = 1{\cdot}5$	2·1181	2·0797	2·0815	2·0995	2·1231
$\alpha = 2{\cdot}0$	2·1221	2·0673	2·0529	2·0590	2·0752
$\alpha = 2{\cdot}5$	2·1303	2·0665	2·0399	2·0353	2·0432
$a = 0{\cdot}6$					
$\alpha = 0{\cdot}5$	2·6097	2·5556	2·5464	2·5504	2·5586
$\alpha = 1{\cdot}0$	2·6380	2·5450	2·5080	2·4956	2·4949
$\alpha = 1{\cdot}5$	2·6729	2·5659	2·5099	2·4812	2·4683
$\alpha = 2{\cdot}0$	2·7029	2·5943	2·5278	2·4874	2·4636
$\alpha = 2{\cdot}5$	2·7272	2·6224	2·5515	2·5035	2·4715
$a = 0{\cdot}8$					
$\alpha = 0{\cdot}5$	3·0908	2·9659	2·9120	2·8850	2·8705
$\alpha = 1{\cdot}0$	3·2007	3·0533	2·9712	2·9213	2·8892
$\alpha = 1{\cdot}5$	3·2696	3·1276	3·0365	2·9746	2·9308
$\alpha = 2{\cdot}0$	3·3143	3·1853	3·0942	3·0277	2·9777
$\alpha = 2{\cdot}5$	3·3473	3·2296	3·1422	3·0752	3·0226
$a = 1{\cdot}0$					
$\alpha = 0{\cdot}5$	3·6416	3·4749	3·3916	3·3083	
$\alpha = 1{\cdot}0$	3·8083	3·6416	3·5416	3·4749	3·4273
$\alpha = 1{\cdot}5$	3·8916	3·7416	3·6416	3·5702	3·5166
$\alpha = 2{\cdot}0$	3·9416	3·8083	3·7130	3·6416	3·5860
$\alpha = 2{\cdot}5$	3·9749	3·8559	3·7666	3·6972	3·6416

the same density, but with parameters (α^*,β^*). Also we assume that ρ_i and θ_i are independently distributed with θ_i uniform over [0, 2π] and similarly the corresponding quantities in the city core, ρ_i^* and θ^*, are independent with θ^* uniform over [0, 2π]. Then, for example, if one takes the route $P_iO_iB_iA_iO^*P^*$ then the total travel distance

$$t = t_{i1} + t_{i1}^* - d_i$$

where t_{i1}^* is the distance associated with C^* corresponding to t_{i1}. Let O_iP_i and O_iB_i cut the ring road, which is at a distance a_i from O_i, at the points T_i and T_i' respectively. Similarly, a ring road at a distance of a^* from O^* be cut by O^*P^* and O^*B_i at T^* and $T^{*\prime}$ respectively. Then some

Table IV.

No.	Travel route	Expected travel distance
1	$P_iO_iB_iA_iO^*P^*$	$(r_i + r^*) + d_i + \frac{\alpha_i r_i}{\alpha_i+\beta_i} + \frac{\alpha^* r^*}{\alpha^*+\beta^*}$
2	$P_iO_iB_iA_iQ^*P$	$(r_i + r^*) + d_i + \frac{\alpha_i r_i}{\alpha_i+\beta_i} + \frac{2}{\pi}\frac{\alpha^* r^*}{\alpha^*+\beta^*}$
3	$P_iO_iB_iA_iP^*$	$r_i + d_i + \frac{\alpha_i r_i}{\alpha_i+\beta_i}$ $+\frac{\alpha^* r^*}{\alpha^*\beta^*}\,{}_3F_2\left(\frac{\alpha^*+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha^*+\beta^*+1}{2}, 1; 1\right)$
4	$P_iO_iB_iP^*$	$r_i + \frac{\alpha_i r_i}{\alpha_i+\beta_i}$ $+R_i^*\frac{\alpha^*}{\alpha^*+\beta^*}\,{}_3F_2\left(\frac{\alpha^*+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha^*+\beta^*+1}{2}, 1; \frac{r^*}{R_i^*}\right)$
5	$P_iO_iB_iA_iT^{*\prime}T^*P^*$	$r_i + d_i + \frac{\alpha_i r_i}{\alpha_i+\beta_i} + r^* + (\pi - 2)a^* + \frac{\alpha^* r^*}{\alpha^*+\beta^*}$ $+2a^*I(\alpha^*, \beta^*; a^*/r^*) - 2r^*\frac{\alpha^*}{\alpha^*+\beta^*}I(\alpha^* + 1, \beta^*; a^*/r^*)$
6	$P_iQ_iB_iA_iO^*P^*$	$d_i + (r_i + r^*) + \frac{2}{\pi}\frac{\alpha_i r_i}{\alpha_i+\beta_i} + \frac{\alpha^* r^*}{\alpha^*+\beta^*}$
7	$P_iQ_iB_iA_iQ^*P^*$	$(r_i + r^*) + d_i + \frac{2}{\pi}\left\{\frac{\alpha_i r_i}{\alpha_i+\beta_i} + \frac{\alpha^* r^*}{\alpha^*+\beta^*}\right\}$
8	$P_iQ_iB_iA_iP^*$	$d_i + r_i + \frac{2}{\pi}\frac{\alpha_i r_i}{\alpha_i+\beta_i}$ $+\frac{\alpha^* r^*}{\alpha^*\beta^*}\,{}_3F_2\left(\frac{\alpha^*+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha^*+\beta^*+1}{2}, 1; 1\right)$
9	$P_iQ_iB_iP^*$	$r_i + \frac{2}{\pi}\frac{\alpha_i r_i}{\alpha_i+\beta_i}$ $+\frac{\alpha^* R_i^*}{\alpha^*+\beta^*}\,{}_3F_2\left(\frac{\alpha^*+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha^*+\beta^*+1}{2}, 1; \frac{r^*}{R_i^*}\right)$
10	$P_iQ_iB_iA_iT^*T^{*\prime}P^*$	$d_i + r_i + \frac{2}{\pi}\frac{\alpha_i r_i}{\alpha_i+\beta_i} + r^* + (\pi - 2)a^* + \frac{\alpha^* r^*}{\alpha^*+\beta^*}$ $+2a^*I(\alpha^*, \beta^*; a^*/r^*) - 2r^*\frac{\alpha^*}{\alpha^*+\beta^*}I(\alpha^* + 1, \beta^*; a^*/r^*)$
11	$P_iB_iA_iO^*P^*$	$d_i + \frac{\alpha_i r_i}{\alpha_i+\beta_i}\,{}_3F_2\left(\frac{\alpha_i+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i-\beta_i+1}{2}, 1; 1\right)$ $+r^* + \frac{\alpha^* r^*}{\alpha^*+\beta^*}$
12	$P_iB_iA_iQ^*P^*$	$d_i + \frac{\alpha_i r_i}{\alpha_i+\beta_i}\,{}_3F_2\left(\frac{\alpha_i+1}{2}; -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i+\beta_i+1}{2}, 1; 1\right)$ $+r^* + \frac{2}{\pi}\frac{\alpha^* r^*}{\alpha^*+\beta^*}$
13	$P_iB_iA_iP^*$	$d_i + \frac{\alpha_i r_i}{\alpha_i+\beta_i}\,{}_3F_2\left(\frac{\alpha_i+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i+\beta_i+1}{2}, 1; 1\right)$ $+\frac{\alpha^* r^*}{\alpha^*+\beta^*}\,{}_3F_2\left(\frac{\alpha^*+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha^*+\beta^*+1}{2}, 1; 1\right)$
14	$P_iB_iP^*$	$\frac{\alpha_i r_i}{\alpha_i+\beta_i}\,{}_3F_2\left(\frac{\alpha_i+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i+\beta_i+1}{2}, 1; 1\right)$ $+\frac{\alpha^* R_i^*}{\alpha^*+\beta^*}\,{}_3F_2\left(\frac{\alpha^*+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha^*+\beta^*+1}{2}, 1; \frac{r^*}{R_i^*}\right)$
15	$P_iB_iA_iT^{*\prime}T^*P^*$	$d_i + \frac{\alpha_i r_i}{\alpha_i+\beta_i}\,{}_3F_2\left(\frac{\alpha_i+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i+\beta_i+1}{2}, 1; 1\right)$ $+r^* + (\pi - 2)a^* + \frac{\alpha^* r^*}{\alpha^*+\beta^*}$ $+2a^*I(\alpha^*, \beta^*; a^*/r^*) - 2r^*\frac{\alpha^*}{\alpha^*+\beta^*}I(\alpha^* + 1, \beta^*; a^*/r^*)$
16	$P_iA_iO^*P^*$	$R_i\frac{\alpha_i}{\alpha_i+\beta_i}\,{}_3F_2\left(\frac{\alpha_i+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i+\beta_i+1}{2}, 1; \frac{r_i}{R_i}\right)$ $+r^* + \frac{\alpha^* r^*}{\alpha^*+\beta^*}$
17	$P_iA_iQ^*P^*$	$R_i\frac{\alpha_i}{\alpha_i+\beta_i}\,{}_3F_2\left(\frac{\alpha_i+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i+\beta_i+1}{2}, 1; \frac{r_i}{R_i}\right)$ $+r^* + \frac{2}{\pi}\frac{\alpha^* r^*}{\alpha^*+\beta^*}$

No.	Travel route	Expected travel distance
18	$P_iA_iP^*$	$R_i\frac{\alpha_i}{\alpha_i+\beta_i}\,{}_3F_2\left(\frac{\alpha_i+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i-\beta_i+1}{2}, 1; \frac{r_i}{R_i}\right)$ $+\frac{\alpha^*r^*}{\alpha^*+\beta^*}\,{}_3F_2\left(\frac{\alpha^*+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha^*+\beta^*+1}{2}, 1; 1\right)$
19	$P_iA_iT^{*\prime}T^*P^*$	$R_i\frac{\alpha_i}{\alpha_i+\beta_i}\,{}_3F_2\left(\frac{\alpha_i+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha_i+\beta_i+1}{2}, 1; \frac{r_i}{R_i}\right)$ $+r^* + (\pi-2)a^* + \frac{\alpha^*r^*}{\alpha^*+\beta^*}$ $+2a^*I(\alpha^*, \beta^*; a^*/r^*) - 2r^*\frac{\alpha^*}{\alpha^*+\beta^*}I(\alpha^*+1, \beta^*; a^*/r^*)$
20	$P_iT_iT_i^{\prime}B_iA_iO^*P^*$	$d_i + (r_i + r^*) + (\pi-2)a_i + \frac{\alpha_ir_i}{\alpha_i+\beta_i} + \frac{\alpha^*r^*}{\alpha^*+\beta^*}$ $+2a_iI(\alpha_i, \beta_i; a_i/r_i) - 2r_i\frac{\alpha_i}{\alpha_i+\beta_i}I(\alpha_i+1, \beta_i; a_i/r_i)$
21	$P_iT_iT_i^{\prime}B_iA_iP^{*\prime}P^*$	$d_i + (r_i + r^*) + (\pi-2)a_i + \frac{\alpha_ir_i}{\alpha_i+\beta_i} + \frac{2}{\pi}\frac{\alpha^*r^*}{\alpha^*+\beta^*}$ $+2a_iI(\alpha_i, \beta_i; a_i/r_i) - 2r_i\frac{\alpha_i}{\alpha_i+\beta_i}I(\alpha_i+1, \beta_i; a_i/r_i)$
22	$P_iT_iT_i^{\prime}B_iA_iP^*$	$d_i + r_i + (\pi-2)a_i + \frac{\alpha_ir_i}{\alpha_i+\beta_i}$ $+2a_iI(\alpha_i, \beta_i; a_i/r_i) - 2r_i\frac{\alpha_i}{\alpha_i+\beta_i}I(\alpha_i+1, \beta_i; a_i/r_i)$ $+\frac{\alpha^*r^*}{\alpha^*+\beta^*}\,{}_3F_2\left(\frac{\alpha^*+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha^*+\beta^*+1}{2}, 1; 1\right)$
23	$P_iT_iT_i^{\prime}B_iP^*$	$r_i + (\pi-2)a_i + \frac{\alpha_ir_i}{\alpha_i+\beta_i}$ $+2a_iI(\alpha_i, \beta_i; a_i/r_i) - 2r_i\frac{\alpha_i}{\alpha_i+\beta_i}I(\alpha_i+1, \beta_i; a_i/r_i)$ $+R_i^*\frac{\alpha^*}{\alpha^*+\beta^*}\,{}_3F_2\left(\frac{\alpha^*+1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{\alpha^*+\beta^*+1}{2}, 1; \frac{r^*}{R_i^*}\right)$
24	$P_iT_iT_i^{\prime}B_iA_iT^{*\prime}T^*P^*$	$d_i + (r_i + r^*) + (\pi-2)(a_i + a^*) + \frac{\alpha_ir_i}{\alpha_i+\beta_i} + \frac{\alpha^*r^*}{\alpha^*+\beta^*}$ $+2a_iI(\alpha_i, \beta_i; a_i/r_i) - 2r_i\frac{\alpha_i}{\alpha_i+\beta_i}I(\alpha_i+1, \beta_i; a_i/r_i)$ $+2a^*I(\alpha^*, \beta^*; a^*/r^*) - 2r^*\frac{\alpha^*}{\alpha^*+\beta^*}I(\alpha^*+1, \beta^*; a^*/r^*)$

routes and the corresponding expected travel distances are given in Table IV where $R_i = r_i + d_i$ and $R_i^* = r^* + d_i$.

One can look at the expected distances for specific values of α_i and α^* and then one can compare the various travel plans by looking at Table IV. The densities and higher order moments of the travel distances can also be computed, but these will take up too much space and hence we will not discuss them here. Expected distances and moments for some special situations are available from Haight (1964) and Wyler (1968).

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