7 Applications of Integration

7.6 Moments, Centers of Mass, and Centroids

Mass

Mass

Several important applications of integration are related to **mass.** Mass is a measure of a body's resistance to changes in motion, and is independent of the particular gravitational system in which the body is located.

However, because so many applications involving mass occur on Earth's surface, an object's mass is sometimes equated with its weight. This is not technically correct.

Weight is a type of force and as such is dependent on gravity. Force and mass are related by the equation

$$\text{Force} = (\text{mass})(\text{acceleration}).$$

Mass

The table below lists some commonly used measures of mass and force, together with their conversion factors.

System of Measurement	Measure of Mass	Measure of Force
U.S.	Slug	Pound = (slug)(ft/sec^2)
International	Kilogram	Newton = (kilogram)(m/sec^2)
C-G-S	Gram	Dyne = (gram)(cm/sec^2)

Conversions:

1 pound = 4.448 newtons	1 slug = 14.59 kilograms
1 newton = 0.2248 pound	1 kilogram = 0.06852 slug
1 dyne = 0.000002248 pound	1 gram = 0.00006852 slug
1 dyne = 0.00001 newton	1 foot = 0.3048 meter

Example 1 – *Mass on the Surface of Earth*

Find the mass (in slugs) of an object whose weight at sea level is 1 pound.

Solution:

Using 32 feet per second per second as the acceleration due to gravity produces

$$\text{Mass} = \frac{\text{force}}{\text{acceleration}} \qquad \text{Force} = (\text{mass})(\text{acceleration})$$

$$= \frac{1 \text{ pound}}{32 \text{ feet per second per second}}$$

$$= 0.03125 \frac{\text{pound}}{\text{foot per second per second}}$$

$$= 0.03125 \text{ slug.}$$

Example 1 – *Solution*

cont'd

Because many applications involving mass occur on Earth's surface, this amount of mass is called a **pound mass.**

Center of Mass in a One-Dimensional System

Center of Mass in a One-Dimensional System

You will now consider two types of moments of a mass—the **moment about a point** and the **moment about a line.** To define these two moments, consider an idealized situation in which a mass *m* is concentrated at a point.

If *x* is the distance between this point mass and another point *P*, the **moment of *m* about the point *P* is**

$$\text{Moment} = mx$$

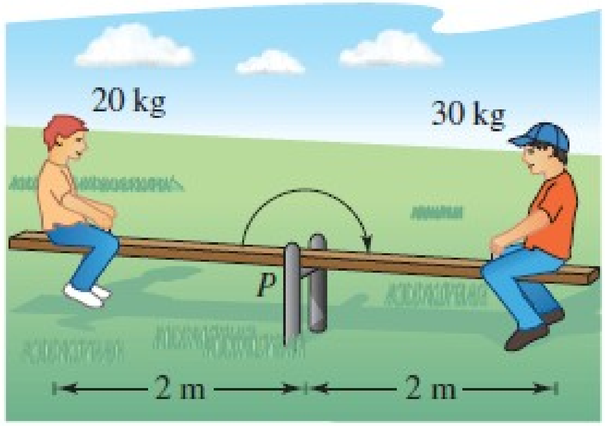
and *x* is the **length of the moment arm.**

Center of Mass in a One-Dimensional System

The concept of moment can be demonstrated simply by a seesaw, as shown in Figure 7.53.

A child of mass 20 kilograms sits 2 meters to the left of the fulcrum *P* , and an older child of mass 30 kilograms sits 2 meters to the right of *P.*

From experience, you know that the seesaw will begin to rotate clockwise, moving the larger child down.



The seesaw will balance when the left and the right moments are equal.

Figure 7.53

Center of Mass in a One-Dimensional System

This rotation occurs because the moment produced by the child on the left is less than the moment produced by the child on the right.

$$\text{Left moment} = (20)(2) = 40 \text{ kilogram-meters}$$

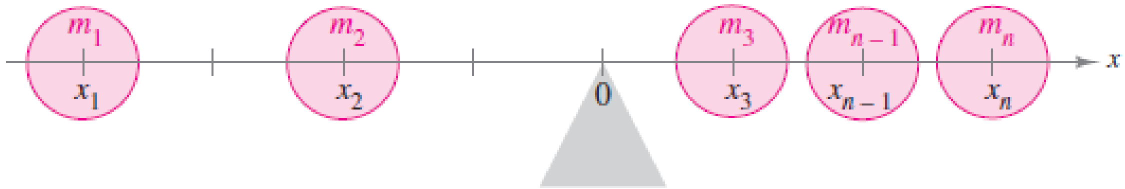
$$\text{Right moment} = (30)(2) = 60 \text{ kilogram-meters}$$

To balance the seesaw, the two moments must be equal. For example, if the larger child moved to a position 4/3 meters from the fulcrum, then the seesaw would balance, because each child would produce a moment of 40 kilogram-meters.

Center of Mass in a One-Dimensional System

You can introduce a coordinate line on which the origin corresponds to the fulcrum, as shown in Figure 7.54.

Several point masses are located on the *x*-axis. The measure of the tendency of this system to rotate about the origin is the **moment about the origin,** and it is defined as the sum of the *n* products $m_i x_i$.



If $m_1x_1 + m_2x_2 + \cdots + m_nx_n = 0$, then the system is in equilibrium.

Figure 7.54

Center of Mass in a One-Dimensional System

The moment about the origin is denoted by M_0 and can be written as

$$M_0 = m_1x_1 + m_2x_2 + \cdot \cdot \cdot + m_nx_n.$$

If M_0 is 0, then the system is said to be in **equilibrium.**

Center of Mass in a One-Dimensional System

For a system that is not in equilibrium, the **center of mass** is defined as the point $\bar{x}$ at which the fulcrum could be relocated to attain equilibrium. If the system were translated $\bar{x}$ units, each coordinate x_i would become $(x_i - \bar{x})$, and because the moment of the translated system is 0, you have

$$\sum_{i=1}^{n} m_i(x_i - \bar{x}) = \sum_{i=1}^{n} m_i x_i - \sum_{i=1}^{n} m_i \bar{x} = 0.$$

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\text{moment of system about origin}}{\text{total mass of system}}.$$

If $m_1x_1 + m_2x_2 + \cdots + m_nx_n = 0$, the system is in equilibrium.

Center of Mass in a One-Dimensional System

Moments and Center of Mass: One-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $x_1, x_2, \ldots, x_n$.

1. The **moment about the origin** is

$$M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n.$$

2. The **center of mass** is

$$\overline{x} = \frac{M_0}{m}$$

where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

Example 2 – *The Center of Mass of a Linear System*

Find the center of mass of the linear system shown in Figure 7.55.

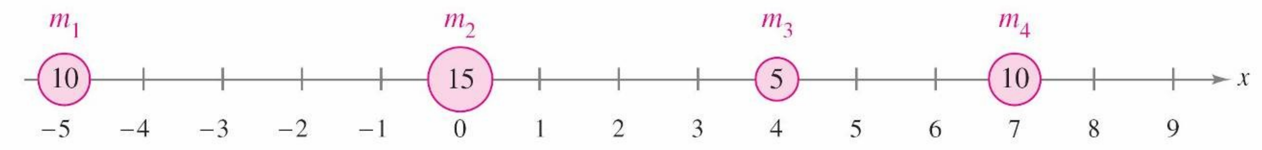


Figure 7.55

Solution:

The moment about the origin is

$M_0 = m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4$

$= 10(–5) + 15(0) + 5(4) + 10(7)$

$= –50 + 0 + 20 + 70$

$= 40.$

Example 2 – *Solution*

cont'd

Because the total mass of the system is

m = 10 + 15 + 5 + 10 = 40, the center of mass is

$$\bar{x} = \frac{M_0}{m} = \frac{40}{40} = 1.$$

Note that the point masses will be in equilibrium when the fulcrum is located at x = 1.

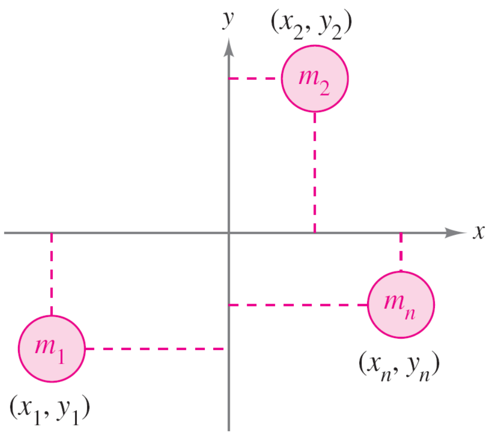
Center of Mass in a Two-Dimensional System

Center of Mass in a Two-Dimensional System

You can extend the concept of moment to two dimensions by considering a system of masses located in the *xy*-plane at the points

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, as shown in Figure 7.56.

Rather than defining a single moment (with respect to the origin), two moments are defined—one with respect to the *x*-axis and one with respect to the *y*-axis.



In a two-dimensional system, there is a moment about the *y*-axis, M_y, and a moment about the *x*-axis, M_x.

Figure 7.56

Center of Mass in a Two-Dimensional System

Moment and Center of Mass: Two-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

1. The **moment about the y-axis** is

$$M_y = m_1x_1 + m_2x_2 + \ldots m_nx_n.$$

2. The **moment about the x-axis** is

$$M_x = m_1y_1 + m_2y_2 + \ldots m_ny_n.$$

3. The **center of mass** $(\overline{x}, \overline{y})$ (or **center of gravity**) is

$$\overline{x} = \frac{M_y}{m} \quad \text{and} \quad \overline{y} = \frac{M_x}{m}$$

where

$$m = m_1 + m_2 + \ldots + m_n$$

is the **total mass** of the system.

Center of Mass in a Two-Dimensional System

The moment of a system of masses in the plane can be taken about any horizontal or vertical line. In general, the moment about a line is the sum of the product of the masses and the *directed distances* from the points to the line.

$$\text{Moment} = m_1(y_1 - b) + m_2(y_2 - b) + \cdot \cdot \cdot + m_n(y_n - b) \quad \text{Horizontal line } y = b$$

$$\text{Moment} = m_1(x_1 - a) + m_2(x_2 - a) + \cdot \cdot \cdot + m_n(x_n - a) \quad \text{Vertical line } x = a$$

Example 3 – *The Center of Mass of a Two-Dimensional System*

Find the center of mass of a system of point masses $m_1 = 6$, $m_2 = 3$, $m_3 = 2$, and $m_4 = 9$, located at (3, –2), (0, 0), (–5, 3), and (4, 2) as shown in Figure 7.57.

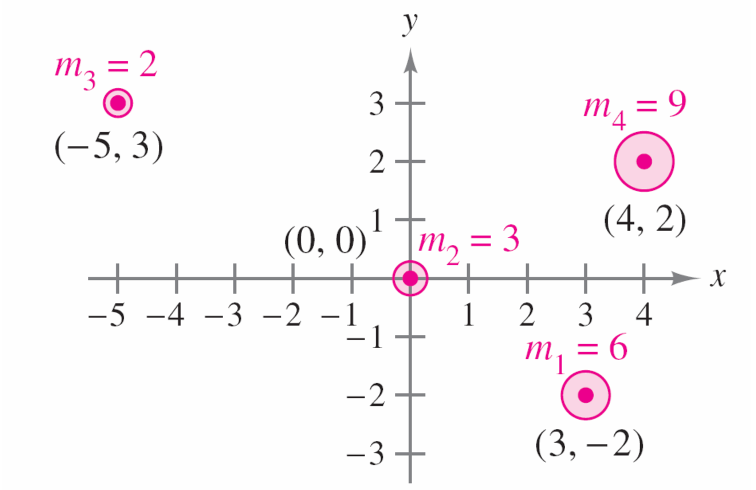


Figure 7.57

Solution:

$$m = 6 + 3 + 2 + 9 = 20 \qquad \text{Mass}$$

$$M_y = 6(3) + 3(0) + 2(-5) + 9(4) = 44 \qquad \text{Moment about } y\text{-axis}$$

$$M_x = 6(-2) + 3(0) + 2(3) + 9(2) = 12 \qquad \text{Moment about } x\text{-axis}$$

Example 3 – *Solution*

cont'd

So,

$$\bar{x} = \frac{M_y}{m} = \frac{44}{20} = \frac{11}{5}$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{12}{20} = \frac{3}{5}$$

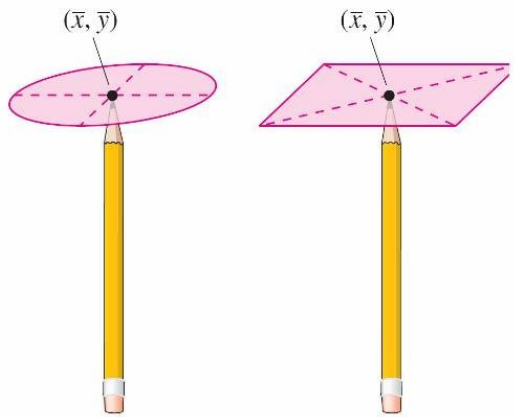
and so the center of mass is $\left(\frac{11}{5}, \frac{3}{5}\right)$.

Center of Mass of a Planar Lamina

Center of Mass of a Planar Lamina

Consider a thin, flat plate of material of constant density called a **planar lamina** (see Figure 7.58).

Density is a measure of mass per unit of volume, such as grams per cubic centimeter. For planar laminas, however, density is considered to be a measure of mass per unit of area. Density is denoted by ρ, the lowercase Greek letter rho.



You can think of the center of mass $(\bar{x}, \bar{y})$ of a lamina as its balancing point. For a circular lamina, the center of mass is the center of the circle. For a rectangular lamina, the center of mass is the center of the rectangle.

Figure 7.58

Center of Mass of a Planar Lamina

Consider an irregularly shaped planar lamina of uniform density ρ, bounded by the graphs of

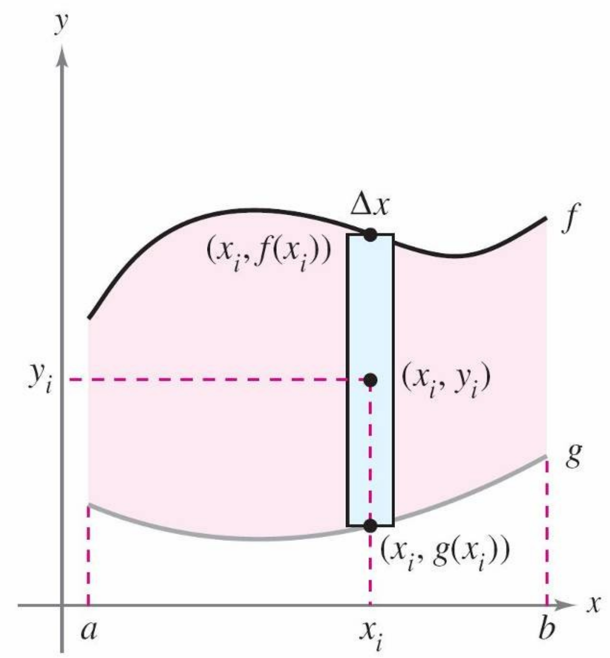
$y = f(x)$, $y = g(x)$, and $a \le x \le b$,

as shown in Figure 7.59.

The mass of this region is given by

$$\begin{aligned} m &= (\text{density})(\text{area}) \\ &= \rho \int_a^b [f(x) - g(x)]\, dx \\ &= \rho A \end{aligned}$$

where A is the area of the region.



Planar lamina of uniform density ρ

Figure 7.59

Center of Mass of a Planar Lamina

To find the center of mass of the lamina, partition the interval [*a, b*] into *n* subintervals of equal width Δx.

Let x_i be the center of the *i*th subinterval. You can approximate the portion of the lamina lying in the *i*th subinterval by a rectangle whose height is $h = f(x_i) - g(x_i)$.

Because the density of the rectangle is ρ, its mass is

$$m_i = (\text{density})(\text{area}) = \underbrace{\rho}_{\text{Density}} \underbrace{[f(x_i) - g(x_i)]}_{\text{Height}} \underbrace{\Delta x}_{\text{Width}}.$$

Center of Mass of a Planar Lamina

Now, considering this mass to be located at the center (x_i, y_i) is $y_i = [f(x_i) + g(x_i)] / 2$. So, the moment of m_i about the x-axis is

$$\begin{aligned}\text{Moment} &= (\text{mass})(\text{distance}) \\ &= m_i y_i \\ &= \rho[f(x_i) - g(x_i)]\,\Delta x\left[\frac{f(x_i) + g(x_i)}{2}\right].\end{aligned}$$

Center of Mass of a Planar Lamina

Moments and Center of Mass of a Planar Lamina

Let f and g be continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, and consider the planar lamina of uniform density ρ bounded by the graphs of $y = f(x)$, $y = g(x)$, and $a \leq x \leq b$.

1. The moments about the x- and y-axes are

$$M_x = \rho \int_a^b \left[\frac{f(x) + g(x)}{2}\right][f(x) - g(x)]\,dx$$

$$M_y = \rho \int_a^b x[f(x) - g(x)]\,dx.$$

2. The center of mass $(\bar{x}, \bar{y})$ is given by $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$, where $m = \rho \int_a^b [f(x) - g(x)]\,dx$ is the mass of the lamina.

Example 4 – *The Center of Mass of a Planar Lamina*

Find the center of mass of the lamina of uniform density ρ bounded by the graph of $f(x) = 4 - x^2$ and the x-axis.

Solution:

Because the center of mass lies on the axis of symmetry, you know that $\bar{x} = 0$.

Moreover, the mass of the lamina is

$$m = \rho \int_{-2}^{2} (4 - x^2)\, dx$$

$$= \rho \left[4x - \frac{x^3}{3} \right]_{-2}^{2}$$

$$= \frac{32\rho}{3}.$$

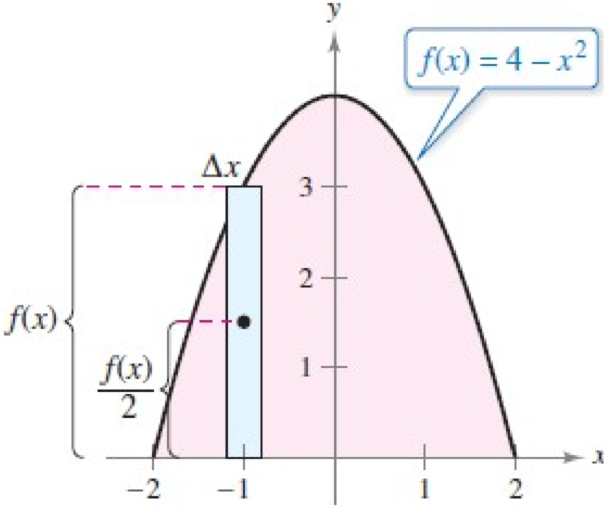
Example 4 – *Solution*

cont'd

To find the moment about the *x*-axis, place a representative rectangle in the region, as shown in the figure below.

The distance from the *x*-axis to the center of this rectangle is

$$y_i = \frac{f(x)}{2} = \frac{4 - x^2}{2}.$$



Example 4 – *Solution*

cont'd

Because the mass of the representative rectangle is

$$\rho f(x)\,\Delta x = \rho(4 - x^2)\,\Delta x$$

you have

$$M_x = \rho \int_{-2}^{2} \frac{4 - x^2}{2}(4 - x^2)\,dx$$

$$= \frac{\rho}{2}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx$$

$$= \frac{\rho}{2}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2}$$

$$= \frac{256\rho}{15}$$

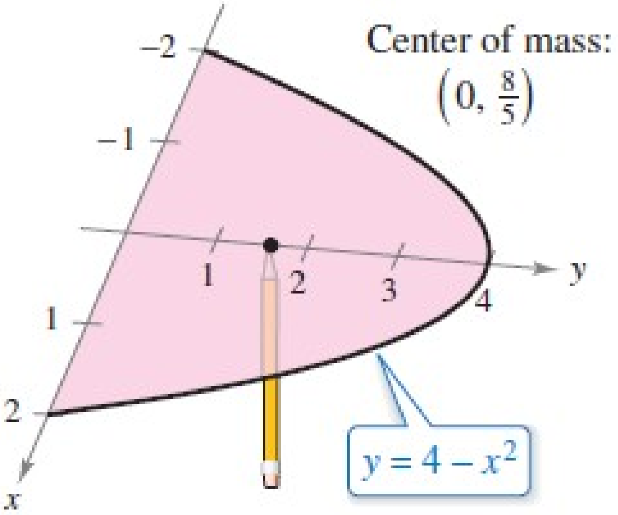
and $\bar{y}$ is given by

$$\bar{y} = \frac{M_x}{m} = \frac{256\rho/15}{32\rho/3} = \frac{8}{5}.$$

Example 4 – *Solution*

cont'd

So, the center of mass (the balancing point) of the lamina is $\left(0, \frac{8}{5}\right)$, as shown in Figure 7.60.



The center of mass is the balancing point.

Figure 7.60

Center of Mass of a Planar Lamina

The center of mass of a lamina of *uniform* density depends only on the shape of the lamina and not on its density. For this reason, the point

$$(\overline{x}, \overline{y}) \qquad \text{Center of mass or centroid}$$

is sometimes called the center of mass of a *region* in the plane, or the **centroid** of the region.

In other words, to find the centroid of a region in the plane, you simply assume that the region has a constant density of $\rho = 1$ and compute the corresponding center of mass.