7 Applications of Integration

7.3 Volume: The Shell Method

Objectives

- Find the volume of a solid of revolution using the shell method.
- Compare the uses of the disk method and the shell method.

The Shell Method

The Shell Method

An alternative method for finding the volume of a solid of revolution is called the **shell method** because it uses cylindrical shells.

A comparison of the advantages of the disk and shell methods is given later in this section.

Consider a representative rectangle as shown in Figure 7.27, where *w* is the width of the rectangle, *h* is the height of the rectangle, and *p* is the distance between the axis of revolution and the *center* of the rectangle.

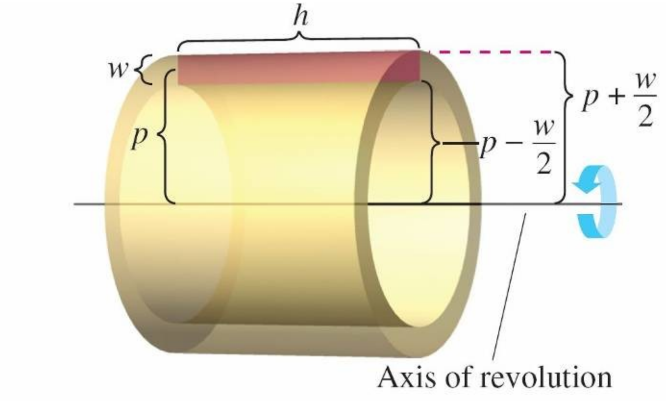


Figure 7.27

The Shell Method

When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness *w.*

To find the volume of this shell, consider two cylinders.

The radius of the larger cylinder corresponds to the outer radius of the shell, and the radius of the smaller cylinder corresponds to the inner radius of the shell. Because *p* is the average radius of the shell, you know the outer radius is

$$p + \frac{w}{2} \qquad \text{Outer radius}$$

and the inner radius is

$$p - \frac{w}{2}. \qquad \text{Inner radius}$$

The Shell Method

So, the volume of the shell is

$$
\begin{aligned}
\text{Volume of shell} &= (\text{volume of cylinder}) - (\text{volume of hole}) \\
&= \pi\left(p + \frac{w}{2}\right)^2 h - \pi\left(p - \frac{w}{2}\right)^2 h \\
&= 2\pi p h w \\
&= 2\pi(\text{average radius})(\text{height})(\text{thickness}).
\end{aligned}
$$

You can use this formula to find the volume of a solid of revolution.

The Shell Method

For instance, the plane region in Figure 7.28 is revolved about a line to form the indicated solid.

Consider a horizontal rectangle of width Δ*y*, then, as the plane region is revolved about a line parallel to the *x*-axis, the rectangle generates a representative shell whose volume is

$$\Delta V = 2\pi[p(y)h(y)]\,\Delta y.$$

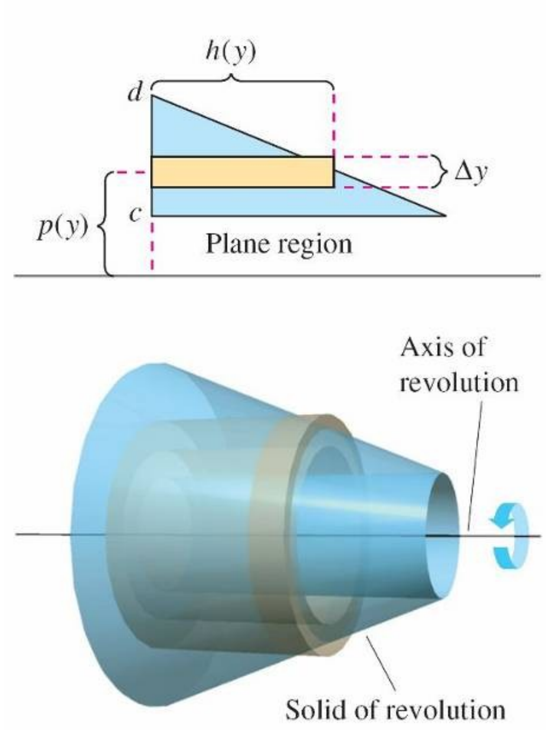


Figure 7.28

The Shell Method

You can approximate the volume of the solid by *n* such shells of thickness Δy, height $h(y_i)$, and average radius $p(y_i)$.

$$\text{Volume of solid} \approx \sum_{i=1}^{n} 2\pi[p(y_i)h(y_i)]\,\Delta y = 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y$$

This approximation appears to become better and better as $\|\Delta\| \to 0 \, (n \to \infty)$. So, the volume of the solid is

$$\begin{aligned}\text{Volume of solid} &= \lim_{\|\Delta\|\to 0} 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y \\ &= 2\pi\int_c^d [p(y)h(y)]\,dy.\end{aligned}$$

The Shell Method

THE SHELL METHOD

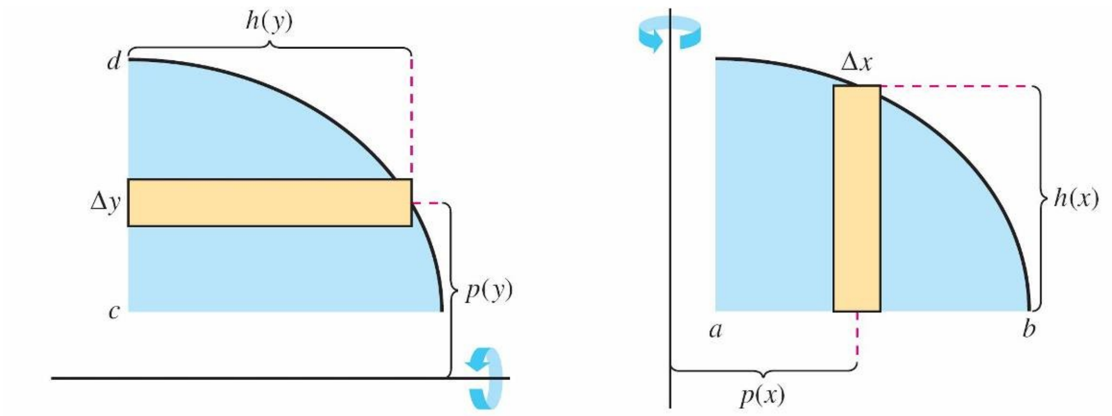
To find the volume of a solid of revolution with the **shell method,** use one of the formulas below. (See Figure 7.29.)

Horizontal Axis of Revolution

$$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\,dy$$

Vertical Axis of Revolution

$$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\,dx$$



Horizontal axis of revolution

Vertical axis of revolution

Figure 7.29

Example 1 – *Using the Shell Method to Find Volume*

Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis $(0 \leq x \leq 1)$ about the y-axis.

Solution:

Because the axis of revolution is vertical, use a vertical representative rectangle, as shown in Figure 7.30. The width Δx indicates that x is the variable of integration.

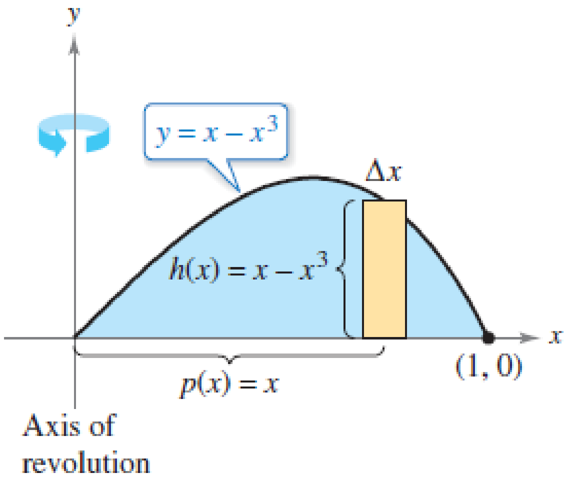


Figure 7.30

Example 1 – *Solution*

cont'd

The distance from the center of the rectangle to the axis of revolution is $p(x) = x$, and the height of the rectangle is

$h(x) = x - x^3$.

Because x ranges from 0 to 1, the volume of the solid is

$$
\begin{aligned}
V &= 2\pi \int_a^b p(x)h(x)\,dx \\
&= 2\pi \int_0^1 x(x - x^3)\,dx \\
&= 2\pi \int_0^1 (-x^4 + x^2)\,dx && \text{Simplify.} \\
&= 2\pi \left[-\frac{x^5}{5} + \frac{x^3}{3} \right]_0^1 && \text{Integrate.} \\
&= 2\pi \left(-\frac{1}{5} + \frac{1}{3} \right) \\
&= \frac{4\pi}{15}
\end{aligned}
$$

Comparison of Disk and Shell Methods

Comparison of Disk and Shell Methods

The disk and shell methods can be distinguished as follows.

For the disk method, the representative rectangle is always *perpendicular* to the axis of revolution, whereas for the shell method, the representative rectangle is always *parallel* to the axis of revolution, as shown in Figure 7.32.

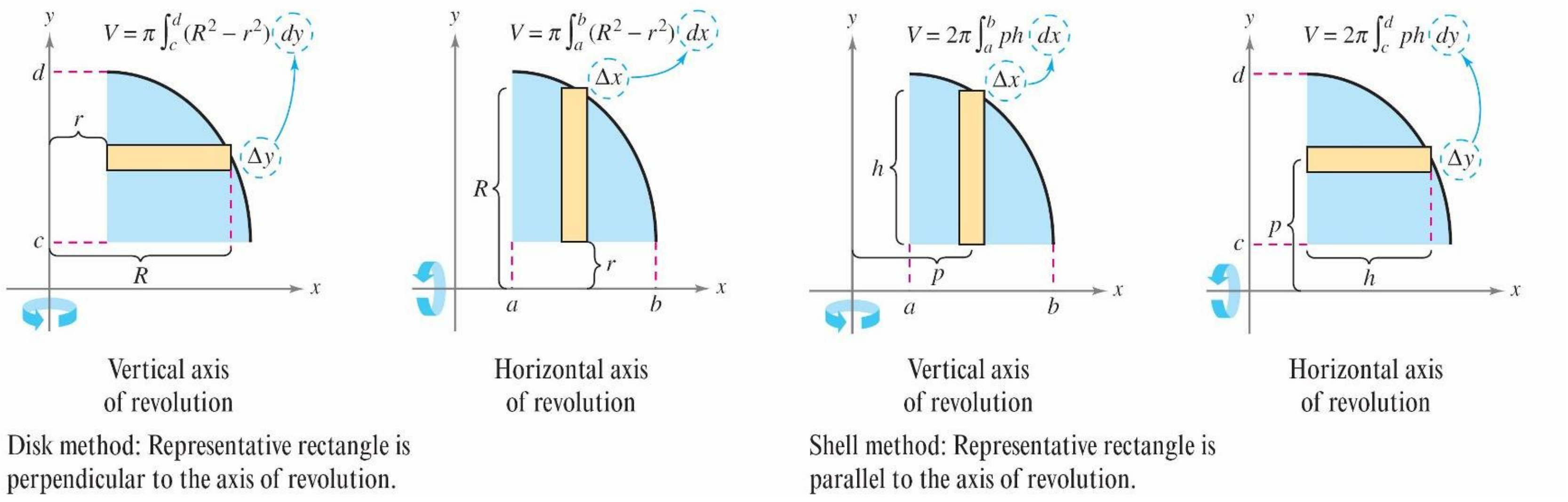


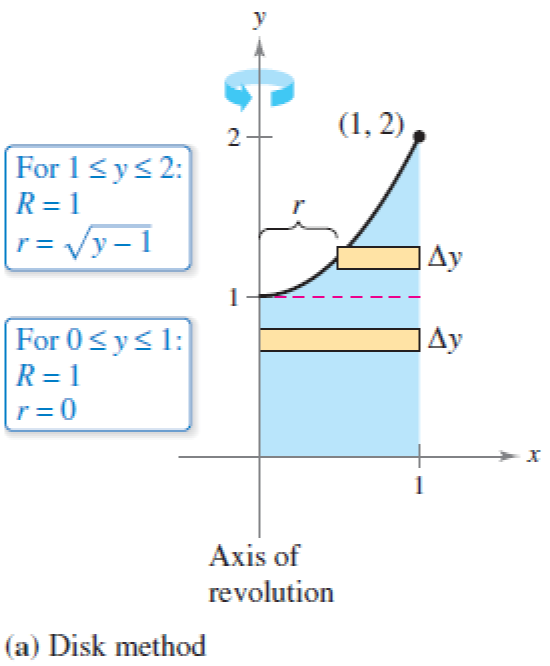
Figure 7.32

Example 3 – *Shell Method Preferable*

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

Solution:

The washer method requires two integrals to determine the volume of this solid. See Figure 7.33(a).



Disk method

Figure 7.33(a)

Example 3 – *Solution*

cont'd

$$V = \pi \int_0^1 (1^2 - 0^2)\, dy + \pi \int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right] dy$$ Apply washer method.

$$= \pi \int_0^1 1\, dy + \pi \int_1^2 (2 - y)\, dy$$ Simplify.

$$= \pi \Big[y \Big]_0^1 + \pi \left[2y - \frac{y^2}{2} \right]_1^2$$ Integrate.

$$= \pi + \pi \left(4 - 2 - 2 + \frac{1}{2} \right)$$

$$= \frac{3\pi}{2}$$

Example 3 – *Solution*

cont'd

In Figure 7.33(b), you can see that the shell method requires only one integral to find the volume.

$$V = 2\pi \int_a^b p(x)h(x)\,dx \qquad \text{Apply shell method.}$$

$$= 2\pi \int_0^1 x(x^2 + 1)\,dx$$

$$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1 \qquad \text{Integrate.}$$

$$= 2\pi\left(\frac{3}{4}\right)$$

$$= \frac{3\pi}{2}$$

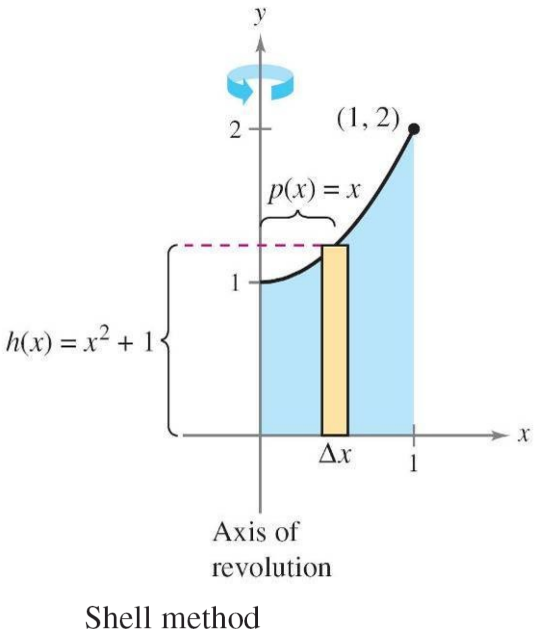


Figure 7.33(b)