7 Applications of Integration

7.4 Arc Length and Surfaces of Revolution

Objectives

- Find the arc length of a smooth curve.
- Find the area of a surface of revolution.

Arc Length

Arc Length

Definite integrals are use to find the arc lengths of curves and the areas of surfaces of revolution.

In either case, an arc (a segment of a curve) is approximated by straight line segments whose lengths are given by the familiar Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

A **rectifiable** curve is one that has a finite arc length.

Arc Length

You will see that a sufficient condition for the graph of a function f to be rectifiable between $(a, f(a))$ and $(b, f(b))$ is that f' be continuous on $[a, b]$.

Such a function is **continuously differentiable** on $[a, b]$, and its graph on the interval $[a, b]$ is a **smooth curve.**

Arc Length

Consider a function $y = f(x)$ that is continuously differentiable on the interval $[a, b]$. You can approximate the graph of f by n line segments whose endpoints are determined by the partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ as shown in Figure 7.37.

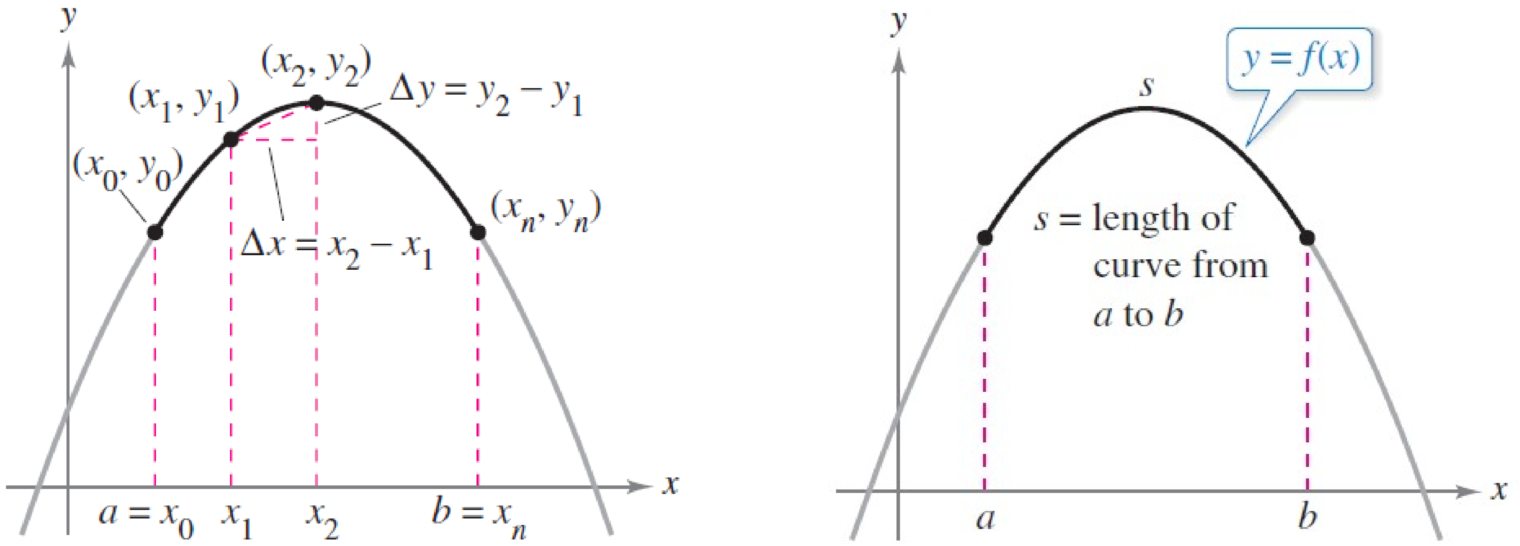


Figure 7.37

Arc Length

By letting $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, you can approximate the length of the graph by

$$\begin{aligned} s &\approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 (\Delta x_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i). \end{aligned}$$

This approximation appears to become better and better as $\|\Delta\| \to 0 \; (n \to \infty)$.

Arc Length

So, the length of the graph is

$$s = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i).$$

Because $f'(x)$ exists for each x in (x_{i-1}, x_i), the Mean Value Theorem guarantees the existence of c_i in (x_{i-1}, x_i) such that

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$$

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i)$$

$$\frac{\Delta y_i}{\Delta x_i} = f'(c_i).$$

Arc Length

Because *f′* is continuous on [*a*, *b*], it follows that $\sqrt{1 + [f'(x)]^2}$ is also continuous (and therefore integrable) on [*a*, *b*], which implies that

$$s = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,(\Delta x_i)$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

where *s* is called the **arc length** of *f* between *a* and *b*.

Arc Length

Definition of Arc Length

Let the function $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

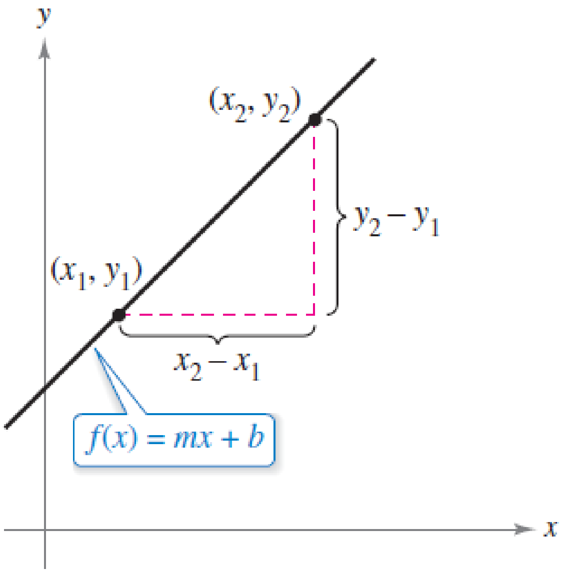
$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Similarly, for a smooth curve $x = g(y)$, the **arc length** of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example 1 – *The Length of a Line Segment*

Find the arc length from (x_1, y_1) to (x_2, y_2) on the graph of $f(x) = mx + b$, as shown in Figure 7.38.



The formula for the arc length of the graph of f from (x_1, y_1) to (x_2, y_2) is the same as the standard Distance Formula.

Figure 7.38

Example 1 – *Solution*

Because

$$m = f'(x) = \frac{y_2 - y_1}{x_2 - x_1}$$

it follows that

$$s = \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\, dx$$ Formula for arc length

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2}\, dx$$

$$= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}\,(x)\Bigg]_{x_1}^{x_2}$$ Integrate and simplify.

Example 1 – *Solution*

cont'd

$$= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}\,(x_2 - x_1)$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which is the formula for the distance between two points in the plane.

Area of a Surface of Revolution

Area of a Surface of Revolution

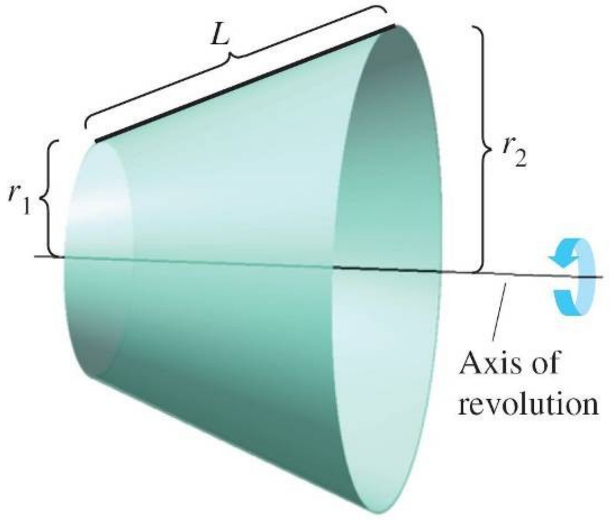
Definition of Surface of Revolution

When the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution.**

The area of a surface of revolution is derived from the formula for the lateral surface area of the frustum of a right circular cone.

Area of a Surface of Revolution

Consider the line segment in the figure below, where L is the length of the line segment, r_1 is the radius at the left end of the line segment, and r_2 is the radius at the right end of the line segment.



Area of a Surface of Revolution

When the line segment is revolved about its axis of revolution, it forms a frustum of a right circular cone, with

$$S = 2\pi r L$$

Lateral surface area of frustum

where

$$r = \frac{1}{2}(r_1 + r_2).$$

Average radius of frustum

Area of a Surface of Revolution

Consider a function *f* that has a continuous derivative on the interval [*a*, *b*]. The graph of *f* is revolved about the *x*-axis to form a surface of revolution, as shown in Figure 7.43.

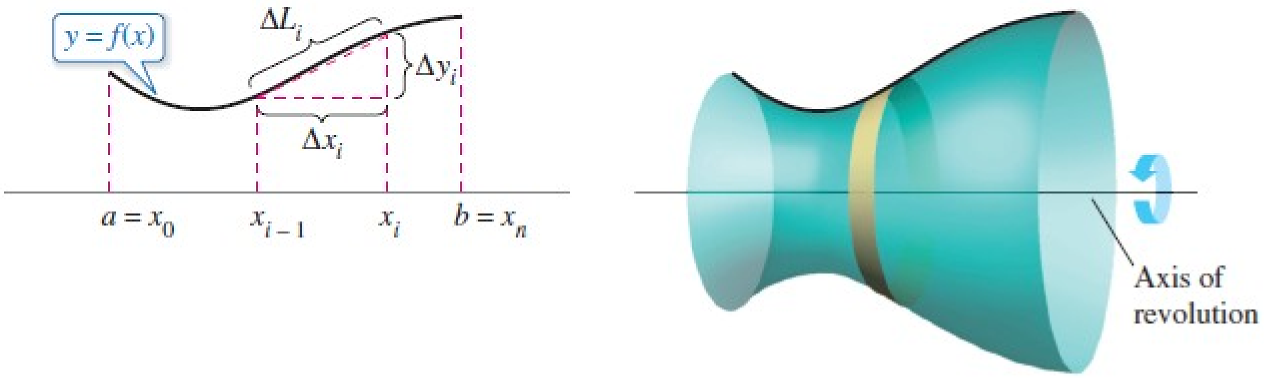


Figure 7.43.

Area of a Surface of Revolution

Let Δ be a partition of [a, b], with subintervals of width Δx_i. Then the line segment of length $\Delta L_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$ generates a frustum of a cone.

Let r_i be the average radius of this frustum.

By the Intermediate Value Theorem, a point d_i exists

(in the ith subinterval) such that $r_i = f(d_i)$.

The lateral surface area ΔS_i of the frustum is

$$\begin{aligned}\Delta S_i &= 2\pi r_i \, \Delta L_i \\ &= 2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} \\ &= 2\pi f(d_i)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i .\end{aligned}$$

Area of a Surface of Revolution

By the Mean Value Theorem, a point c_i exists in (x_{i-1}, x_i) such that

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x_i}.$$

So, $\Delta S_i = 2\pi f(d_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x_i,$ and the total surface area

can be appro: $$S \approx 2\pi \sum_{i=1}^{n} f(d_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x_i.$$

Area of a Surface of Revolution

It can be shown that the limit of the right side as $\|\Delta\| \to 0 \ (n \to \infty)$ is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

In a similar manner, if the graph of *f* is revolved about the *y*-axis, then *S* is

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\, dx.$$

Area of a Surface of Revolution

In these two formulas for S, you can regard the products $2\pi f(x)$ and $2\pi x$ as the circumferences of the circles traced by a point (x, y) on the graph of f as it is revolved about the x-axis and the y-axis (Figure 7.44). In one case the radius is $r = f(x)$, and in the other case the radius is $r = x$.

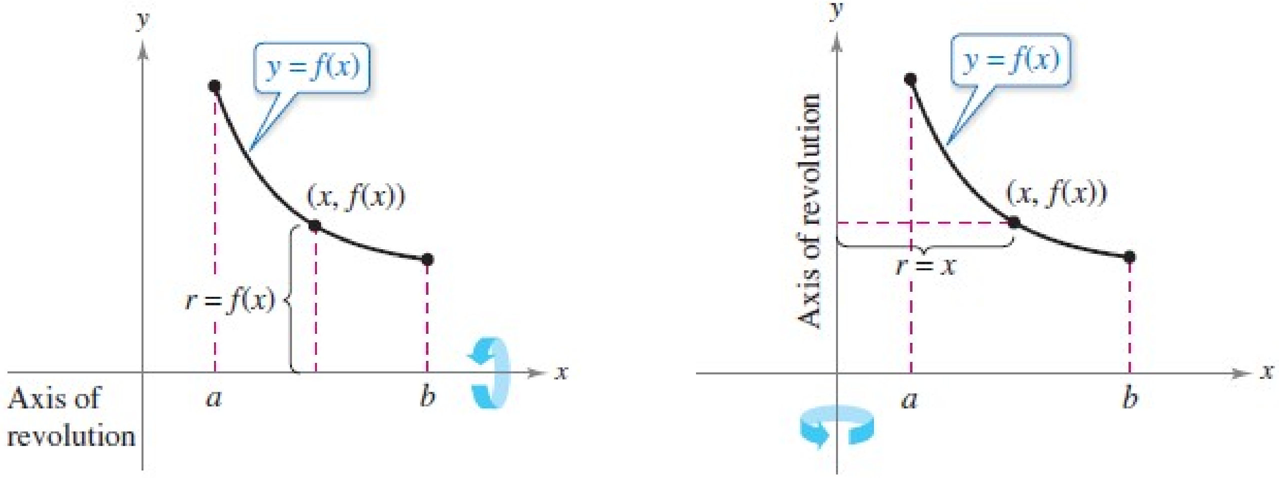


Figure 7.44

Area of a Surface of Revolution

Definition of the Area of a Surface of Revolution

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx \qquad y \text{ is a function of } x.$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\, dy \qquad x \text{ is a function of } y.$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

Example 6 – *The Area of a Surface of Revolution*

Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval [0, 1] about the x-axis, as shown in Figure 7.45.

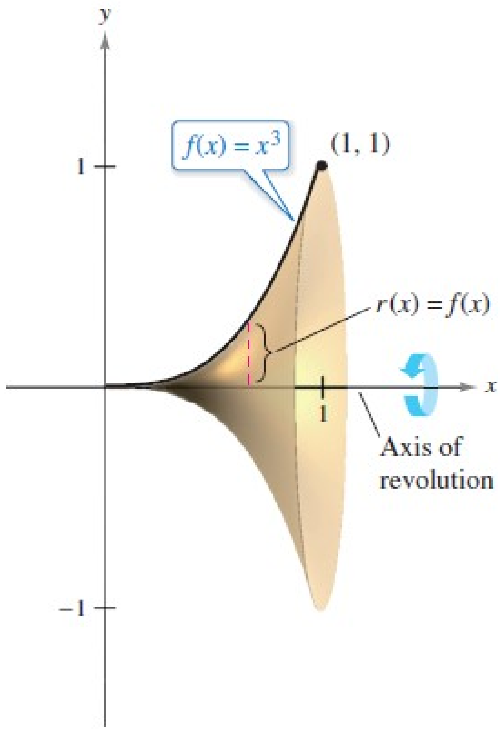


Figure 7.45

Example 6 – *Solution*

The distance between the x-axis and the graph of f is $r(x) = f(x)$, and because $f'(x) = 3x^2$, the surface area is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx \qquad \text{Formula for surface area}$$

$$= 2\pi \int_0^1 x^3 \sqrt{1 + (3x^2)^2}\, dx$$

$$= \frac{2\pi}{36} \int_0^1 (36x^3)(1 + 9x^4)^{1/2}\, dx \qquad \text{Simplify.}$$

$$= \frac{\pi}{18}\left[\frac{(1 + 9x^4)^{3/2}}{3/2}\right]_0^1 \qquad \text{Integrate.}$$

$$= \frac{\pi}{27}(10^{3/2} - 1)$$

$$\approx 3.563.$$