7 Applications of Integration

7.5 Work

Objectives

- Find the work done by a constant force.
- Find the work done by a variable force.

Work Done by a Constant Force

Work Done by a Constant Force

In general, **work** is done by a force when it moves an object. If the force applied to the object is *constant,* then the definition of work is as follows.

Definition of Work Done by a Constant Force

If an object is moved a distance D in the direction of an applied constant force F, then the **work** W done by the force is defined as $W = FD$.

Work Done by a Constant Force

- There are many types of forces—gravitational, electromagnetic, strong nuclear, and weak nuclear.

- A **force** can be thought of as a *push* or a *pull*; a force changes the state of rest or state of motion of a body.

- For gravitational forces on Earth, it is common to use units of measure corresponding to the weight of an object.

Example 1 – *Lifting an Object*

Determine the work done in lifting a 50-pound object 4 feet.

Solution:

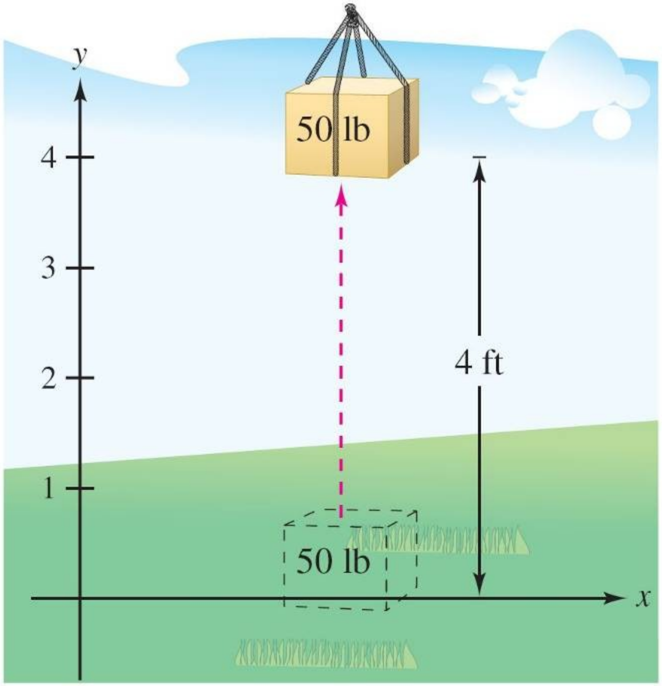
The magnitude of the required force F is the weight of the object, as shown in Figure 7.46.

So, the work done in lifting the object 4 feet is

$W = FD$ Work = (force)(distance)

$= 50(4)$ Force = 50 pounds, distance = 4 feet

$= 200$ foot-pounds



The work done in lifting a 50-pound object 4 feet is 200 foot-pounds.

Figure 7.46

Work Done by a Constant Force

In the U.S. measurement system, work is typically expressed in foot-pounds (ft-lb), inch-pounds, or foot-tons.

In the International System of Units (SI), the basic unit of force is the **newton** – the force required to produce an acceleration of 1 meter per second per second on a mass of 1 kilogram. In this system, work is typically expressed in newton-meters, also called joules.

Work Done by a Constant Force

In another system, the centimeter-gram-second (C-G-S) system, the basic unit of force is the **dyne**—the force required to produce an acceleration of 1 centimeter per second per second on a mass of 1 gram.

In this system, work is typically expressed in dyne-centimeters (ergs) or newton-meters (joules), where 1 joule = 10^7 ergs.

Work Done by a Variable Force

Work Done by a Variable Force

When a *variable* for is applied to an object, calculus is needed to determine the work done, because the amount of force changes as the object changes position.

For instance, the force required to compress a spring increases as the spring is compressed.

Work Done by a Variable Force

Consider an object that is moved along a straight line from $x = a$ to $x = b$ by a continuously varying force $F(x)$.

Let Δ be a partition that divides the interval [a, b] into n subintervals determined by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b$$

and let $\Delta x_i = x_i - x_{i-1}$. For each i , choose c_i such that

$$x_{i-1} \le c_i \le x_i.$$

Work Done by a Variable Force

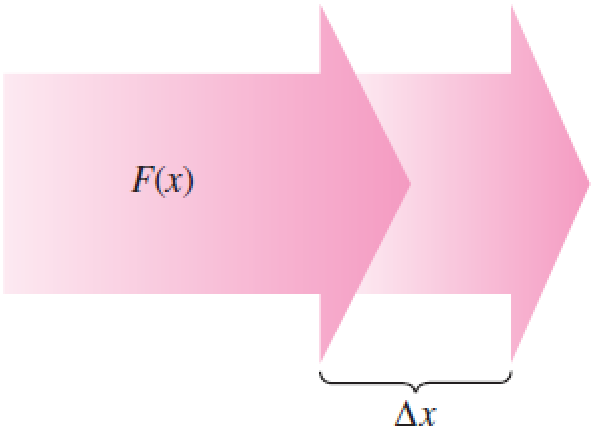
Then at c_i , the force is $F(c_i)$. Because F is continuous, you can approximate the work done in moving the object through the ith subinterval by the increment

$$\Delta W_i = F(c_i)\,\Delta x_i$$

as shown in Figure 7.47. So, the total work done as the object moves from a to b is approximated

$$W \approx \sum_{i=1}^{n} \Delta W_i$$

$$= \sum_{i=1}^{n} F(c_i)\,\Delta x_i.$$



The amount of force changes as an object changes position (Δx).

Figure 7.47

Work Done by a Variable Force

This approximation appears to become better and better as $\|\Delta\| \to 0 \ (n \to \infty)$. So, the work done is

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} F(c_i)\, \Delta x_i$$

$$= \int_a^b F(x)\, dx.$$

Work Done by a Variable Force

Definition of Work Done by a Variable Force

If an object is moved along a straight line by a continuously varying force $F(x)$, then the **work** W done by the force as the object is moved from

$$x = a \quad \text{to} \quad x = b$$

is given by

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta W_i$$
$$= \int_a^b F(x)\,dx.$$

Work Done by a Variable Force

The following three laws of physics were developed by Robert Hooke (1635–1703), Isaac Newton (1642–1727), and Charles Coulomb (1736 –1806).

1. **Hooke's Law:** The force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is,

$$F = kd$$

where the constant of proportionality k (the spring constant) depends on the specific nature of the spring.

Work Done by a Variable Force

2. Newton's Law of Universal Gravitation: The force *F* of attraction between two particles of masses m_1 and m_2 is proportional to the product of the masses and inversely proportional to the square of the distance *d* between the two particles. That is,

$$F = G\frac{m_1 m_2}{d^2}.$$

If m_1 and m_2 are given in grams and *d* in centimeters, *F* will be in newtons for a value of

$G = 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared, where *G* is the **gravitational constant.**

Work Done by a Variable Force

3. Coulomb's Law: The force F between two charges q_1 and q_2 in a vacuum is proportional to the product of the charges and inversely proportional to the square of the distance d between the two charges. That is,

$$F = k\,\frac{q_1 q_2}{d^2}.$$

If q_1 and q_2 are given in electrostatic units and d in centimeters, F will be in dynes for a value of $k = 1$.

Example 2 – *Compressing a Spring*

A force of 750 pounds compresses a spring 3 inches from its natural length of 15 inches. Find the work done in compressing the spring an additional 3 inches.

Solution:

By Hooke's Law, the force $F(x)$ required to compress the spring x units (from its natural length) is $F(x) = kx$.

Because $F(3) = 750$, it follows that

$$F(3) = (k)(3) \Rightarrow 750 = 3k \Rightarrow 250 = k.$$

So, $F(x) = 250x$, as shown in Figure 7.48.

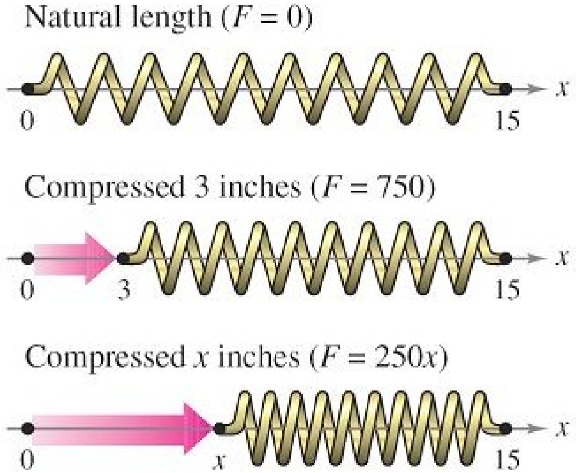


Figure 7.48

Example 2 – *Solution*

cont'd

To find the increment of work, assume that the force required to compress the spring over a small increment Δx is nearly constant.

So, the increment of work is

ΔW = (force) (distance increment) = $(250x)\ \Delta x$.

Because the spring is compressed from $x = 3$ to $x = 6$ inches less than its natural length, the work required is

$$W = \int_a^b F(x)\,dx = \int_3^6 250x\,dx = 125x^2\Big]_3^6$$

$$= 4500 - 1125 = 3375 \text{ inch-pounds.}$$

Example 2 – *Solution*

cont'd

Note that you do *not* integrate from $x = 0$ to $x = 6$ because you were asked to determine the work done in compressing the spring an *additional* 3 inches (not including the first 3 inches).

Work Done by a Variable Force

$$\Delta W = (\text{force})(\text{distance increment}) = (F)(\Delta x).$$

Another way to formulate the increment of work is

$$\Delta W = (\text{force increment})(\text{distance}) = (\Delta F)(x).$$

This second interpretation of ΔW is useful in problems involving the movement of nonrigid substances such as fluids and chains.

Example 4 – *Emptying a Tank of Oil*

A spherical tank of radius 8 feet is half full of oil that weighs 50 pounds per cubic foot. Find the work required to pump oil out through a hole in the top of the tank.

Solution:

Consider the oil to be subdivided into disks of thickness Δy and radius x, as shown in Figure 7.50.

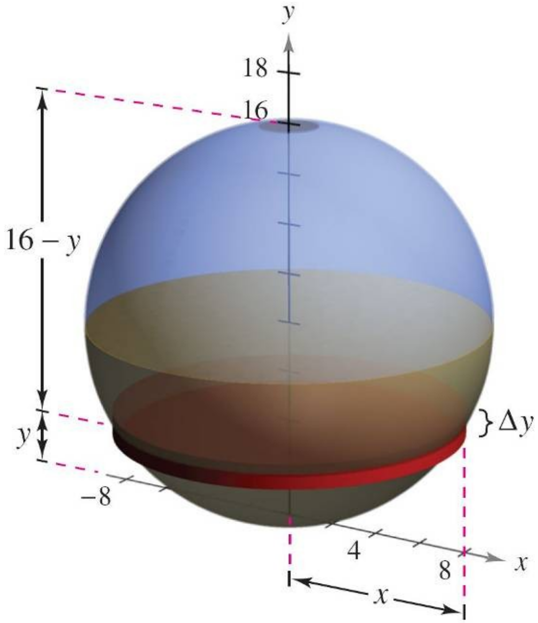


Figure 7..50

Example 4 – *Solution*

cont'd

Because the increment of force for each disk is given by its weight, you have

$$\begin{aligned}\Delta F &= \text{weight}\\ &= \left(\frac{50 \text{ pounds}}{\text{cubic foot}}\right)(\text{volume})\\ &= 50(\pi x^2 \Delta y) \ \text{ pounds.}\end{aligned}$$

For a circle of radius 8 and center at (0, 8), you have

$$x^2 + (y - 8)^2 = 8^2$$

$$x^2 = 16y - y^2$$

and you can write the force increment as

$$\begin{aligned}\Delta F &= 50(\pi x^2 \Delta y)\\ &= 50\pi(16y - y^2)\Delta y.\end{aligned}$$

Example 4 – *Solution*

cont'd

In Figure 7.50, note that a disk y feet from the bottom of the tank must be moved a distance of $(16 - y)$ feet.

So, the increment of work is

$$\Delta W = \Delta F(16 - y)$$
$$= 50\pi(16y - y^2)\Delta y(16 - y)$$
$$= 50\pi(256y - 32y^2 + y^3)\Delta y.$$

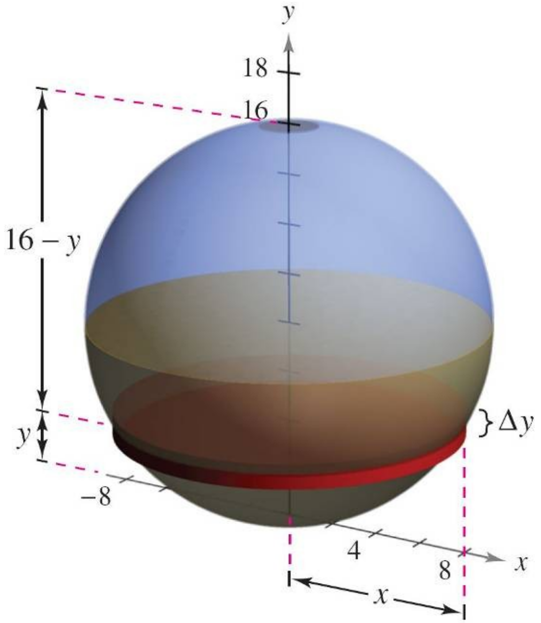


Figure 7.50

Example 4 – *Solution*

cont'd

Because the tank is half full, *y* ranges from 0 to 8, and the work required to empty the tank is

$$
\begin{aligned}
W &= \int_0^8 50\pi(256y - 32y^2 + y^3)\,dy \\
&= 50\pi\left[128y^2 - \frac{32}{3}y^3 + \frac{y^4}{4}\right]_0^8 \\
&= 50\pi\left(\frac{11{,}264}{3}\right) \\
&\approx 589{,}782 \text{ foot-pounds.}
\end{aligned}
$$

Example 4 – *Solution*

To estimate the reasonableness of the result in Example 4, consider that the weight of the oil tank is

$$\left(\frac{1}{2}\right)(\text{volume})(\text{density}) = \frac{1}{2}\left(\frac{4}{3}\pi 8^3\right)(50) \approx 53{,}616.5 \text{ pounds}$$

Lifting the entire half-tank of oil 8 feet would involve work of

$$\begin{aligned} W &= FD && \text{Formula for work done by a constant force} \\ &\approx (53{,}616.5)(8) \\ &= 428{,}932 \text{ foot-pounds.} \end{aligned}$$

Because the oil is actually lifted between 8 and 16 feet, it seems reasonable that the work done is about 589,782 foot-pounds.