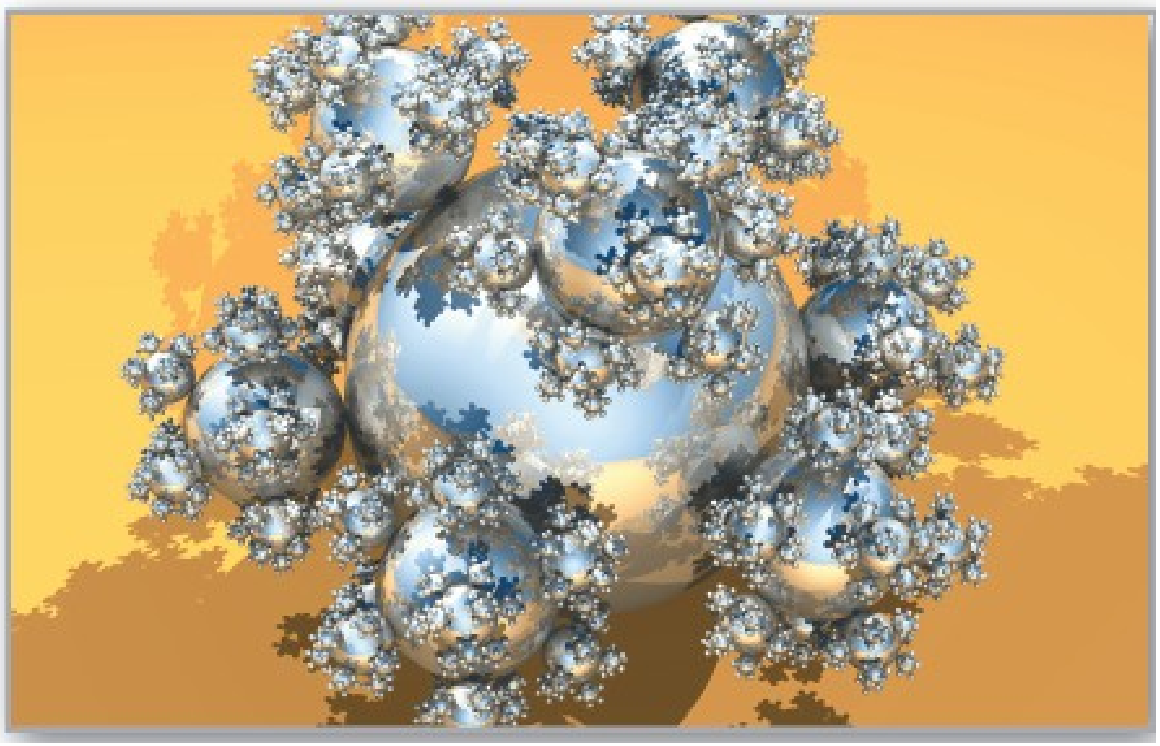
9 Infinite Series



9.10 Taylor and Maclaurin Series

Objectives

- Find a Taylor or Maclaurin series for a function.
- Find a binomial series.
- Use a basic list of Taylor series to find other Taylor series.

Taylor Series and Maclaurin Series

Taylor Series and Maclaurin Series

The next theorem gives the form that *every* convergent power series must take.

THEOREM 9.22 The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \Sigma\, a_n(x - c)^n$ for all x in an open interval I containing c, then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots.$$

The coefficients of the power series in Theorem 9.22 are precisely the coefficients of the Taylor polynomials for $f(x)$ at c. For this reason, the series is called the **Taylor series** for $f(x)$ at c.

Taylor Series and Maclaurin Series

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at c**. Moreover, if $c = 0$, then the series is the **Maclaurin series for f**.

Example 1 – *Forming a Power Series*

Use the function $f(x) = \sin x$ to form the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdot \cdot \cdot$$

and determine the interval of convergence.

Solution:

Successive differentiation of $f(x)$ yields

$f(x) = \sin x$ $\qquad$ $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\qquad$ $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $\qquad$ $f''(0) = -\sin 0 = 0$

$f^{(3)}(x) = -\cos x$ $\qquad$ $f^{(3)}(0) = -\cos 0 = -1$

Example 1 – *Solution*

cont'd

$f^{(4)}(x) = \sin x$ $\qquad$ $f^{(4)}(0) = \sin 0 = 0$

$f^{(5)}(x) = \cos x$ $\qquad$ $f^{(5)}(0) = \cos 0 = 1$

and so on.

The pattern repeats after the third derivative.

Example 1 – *Solution*

cont'd

So, the power series is as follows.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &= 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 \\ &\qquad + \frac{(-1)}{7!}x^7 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned}$$

By the Ratio Test, you can conclude that this series converges for all *x*.

Taylor Series and Maclaurin Series

You cannot conclude that the power series converges to sin *x* for all *x*.

You can simply conclude that the power series converges to some function, but you are not sure what function it is.

This is a subtle, but important, point in dealing with Taylor or Maclaurin series.

To persuade yourself that the series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

might converge to a function other than *f*, remember that the derivatives are being evaluated at a single point.

Taylor Series and Maclaurin Series

It can easily happen that another function will agree with the values of $f^{(n)}(x)$ when $x = c$ and disagree at other x-values.

If you formed the power series for the function shown in Figure 9.23, you would obtain the same series as in Example 1.

You know that the series converges for all x, and yet it obviously cannot converge to both $f(x)$ and $\sin x$ for all x .

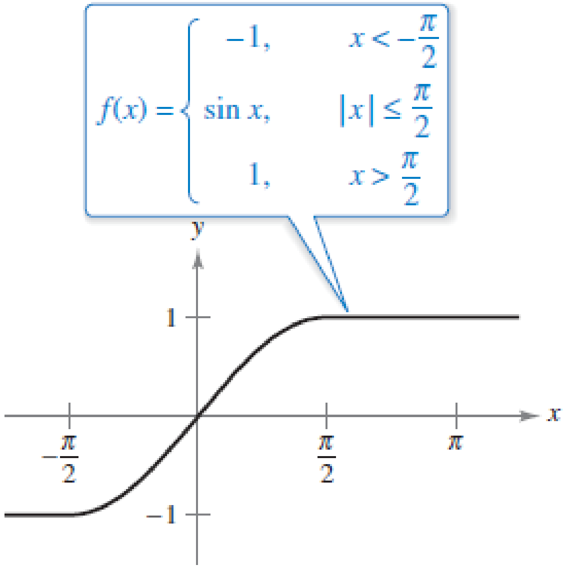


Figure 9.23

Taylor Series and Maclaurin Series

Let *f* have derivatives of all orders in an open interval *I* centered at *c*.

The Taylor series for *f* may fail to converge for some *x* in *I*. Or, even if it is convergent, it may fail to have *f*(*x*) as its sum.

Nevertheless, Theorem 9.19 tells us that for each *n*,

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}.$$

Taylor Series and Maclaurin Series

Note that in this remainder formula, the particular value of *z* that makes the remainder formula true depends on the values of *x* and *n*. If $R_n \to 0$, then the next theorem tells us that the Taylor series for *f* actually converges to *f* (*x*) for all *x* in *I*.

THEOREM 9.23 Convergence of Taylor Series

If $\lim_{n \to \infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

Example 2 – *A Convergent Maclaurin Series*

Show that the Maclaurin series for $f(x) = \sin x$ converges to $\sin x$ for all x.

Solution:

You need to show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

is true for all x.

Example 2 – *Solution*

cont'd

Because

$$f^{(n+1)}(x) = \pm \sin x$$

or

$$f^{(n+1)}(x) = \pm \cos x$$

you know that $\left| f^{(n+1)}(z) \right| \le 1$ for every real number *z.*

Therefore, for any fixed *x*, you can apply Taylor's Theorem (Theorem 9.19) to conclude that

$$0 \le \left| R_n(x) \right| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \le \frac{\left| x \right|^{n+1}}{(n+1)!}.$$

Example 2 – *Solution*

cont'd

From the discussion regarding the relative rates of convergence of exponential and factorial sequences, it follows that for a fixed x

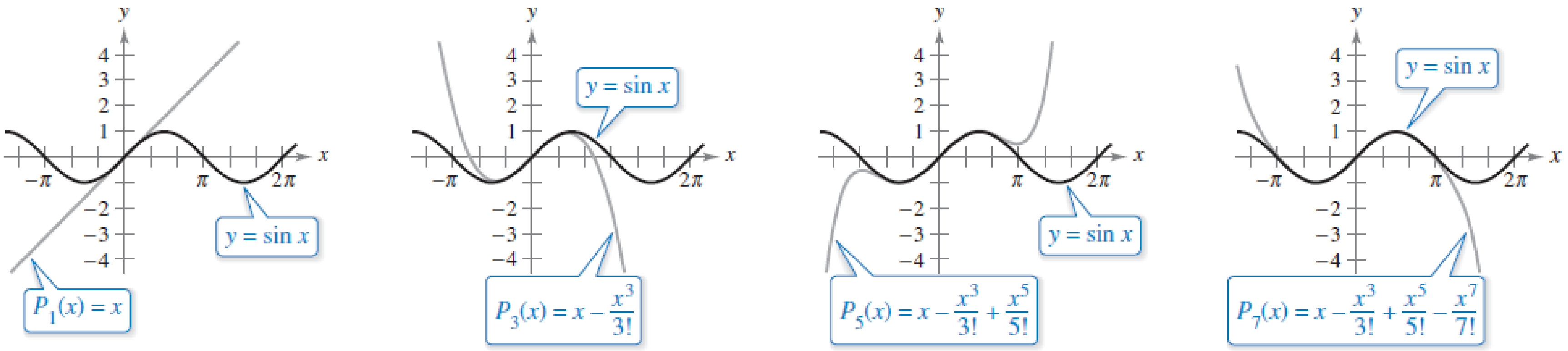
$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Finally, by the Squeeze Theorem, it follows that for all x, $R_n(x) \to 0$ as $n \to \infty$.

So, by Theorem 9.23, the Maclaurin series for sin x converges to sin x for all x.

Taylor Series and Maclaurin Series

Figure 9.24 visually illustrates the convergence of the Maclaurin series for sin *x* by comparing the graphs of the Maclaurin polynomials $P_1(x)$, $P_3(x)$, $P_5(x)$, and $P_7(x)$ with the graph of the sine function. Notice that as the degree of the polynomial increases, its graph more closely resembles that of the sine function.



As n increases, the graph of P_n more closely resembles the sine function.

Figure 9.24

Taylor Series and Maclaurin Series

GUIDELINES FOR FINDING A TAYLOR SERIES

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c), f'(c), f''(c), f'''(c), \cdot \cdot \cdot, f^{(n)}(c), \cdot \cdot \cdot$$

Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdot \cdot \cdot .$$

3. Within this interval of convergence, determine whether the series converges to $f(x)$.

Binomial Series

Binomial Series

Before presenting the basic list for elementary functions, you will develop one more series—for a function of the form $f(x) = (1 + x)^k$. This produces the **binomial series.**

Example 4 – *Binomial Series*

Find the Maclaurin series for $f(x) = (1 + x)^k$ and determine its radius of convergence.

Assume that k is not a positive integer and $k \neq 0$.

Solution:

By successive differentiation, you have

$$f(x) = (1 + x)^k \qquad f(0) = 1$$

$$f'(x) = k(1 + x)^{k-1} \qquad f'(0) = k$$

$$f''(x) = k(k - 1)(1 + x)^{k-2} \qquad f''(0) = k(k - 1)$$

$$f'''(x) = k(k - 1)(k - 2)(1 + x)^{k-3} \qquad f'''(0) = k(k - 1)(k - 2)$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = k\cdots(k - n + 1)(1 + x)^{k-n} \qquad f^{(n)}(0) = k(k - 1)\cdots(k - n + 1)$$

Example 4 – *Binomial Series*

cont'd

which produces the series

$$1 + kx + \frac{k(k-1)x^2}{2} + \cdots + \frac{k(k-1)\cdots(k-n+1)x^n}{n!} + \cdots.$$

Because $a_{n+1}/a_n \to 1$, you can apply the Ratio Test to conclude that the radius of convergence is $R = 1$.

So, the series converges to some function in the interval $(-1, 1)$.

Deriving Taylor Series from a Basic List

Deriving Taylor Series from a Basic List

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \leq x \leq 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1^*$

* The convergence at $x = \pm 1$ depends on the value of k.

Example 6 – *Deriving a Power Series from a Basic List*

Find the power series for $f(x) = \cos\sqrt{x}$.

Solution:

Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

you can replace *x* by $\sqrt{x}$ to obtain the series

$$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges for all *x* in the domain of $\cos\sqrt{x}$—that is, for $x \geq 0$.