CPS 5310 System of ODEs

Consider a mass m attached to an end of a spiral spring whose other end is fixed to a support as shown in Figure 1 below. The body is placed on a smooth horizontal surface. Let the mass be displaced through a distance x towards right and released. It will oscillate about its mean position. The restoring force F acts in the opposite direction and is proportional to the displacement x with constant of proportionality $k > 0$, we call it the spring constant.

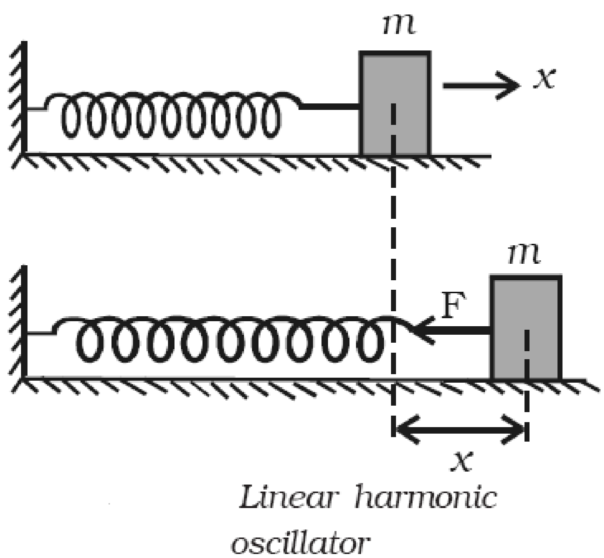

Figure 1: Description of the harmonic oscillator

Newton's Second Law of motion states tells us that the acceleration of an object due to an applied force is in the direction of the force and inversely proportional to the mass being moved. This can be expressed in the form

$$F_{\text{net}} = m\frac{d^2x}{dt^2} \tag{1}$$

where the force $F_{\text{net}} = F + F_f$ consists of

I. The restoring force $F = -kx,\ k > 0$, and

II. Friction $F_f(x) = -b\ dx/dt$, which is a force opposing motion and we assume that it is proportional to velocity with constant of proportionality $b \geq 0$ called the damping coefficient.

Assuming there are no other forces acting on the system (1), we have a **Harmonic Oscillator** which can be expressed as the following second order system

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \tag{2}$$

For the Harmonic Oscillator (2), suppose the mass $m = 1$, the spring constant $b = 7$ and the damping coefficient $k = 10$.

1. With the parameter values of m, b, k as specified above, please express the given second order system (2), as a system of two first-order equations

$$\frac{d\mathbf{X}}{dt} = A\mathbf{X}, \tag{3}$$

   please clearly describe $\mathbf{X}$ and A.

2. What is the equilibrium displacement of this system?

3. Compute the eigenvalues and associated eigenvectors and determine the general solution involving unknown constants k_1 and k_2.

4. Please sketch the phase portrait for the system (3).

5. Based on the phase portrait obtained in (c) please remark on the long term behavior of the displacement $x(t)$ i.e., does $x(t)$ converge the equilibrium displacement as $t \to \infty$?

   And is this convergence independent of initial displacement and initial velocity?