Tuesday, January 28, 2020 4:34 PM

Notation: Error Bounds

$$|f(x) - p_n(x)| \le \boxed{} \text{ for all } -\frac{\pi}{4} \le x \le \pi/4.$$

True value — approximated valued

Worse possible bound for the error $|f(x) - p_n(x)|$

Taylor series of $f(x)$ centered at a.

$$f(x) = f(a) + f'(a)(x-a) + \cdots \frac{f^{(n)}(a)(x-a)^n}{n!} + \cdots$$

e.g: $f(x) = \cos x$, $a = 0$

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$$

$f(x) = \cos x$

$f^{(2n+1)}(x) = (-1)^{2n+1} \sin x \rightarrow$

$f^{(2n)}(x) = (-1)^n x^{2n}/(2n)!$

$x = a = 0$

$$\cos x = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \cdots$$

$$= \cos 0 + \sin 0 \, x + (-\cos 0)\, x^2/2 + \cdots$$

$$= 1 + 0 - x^2/2 + \cdots \text{ (Maclaurin Series)}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x)$$

Remainder Term

Remainder Term

$$\cos x = \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots (-1)^n \frac{x^{2n}}{(2n)!}}_{P_{2n}(x)} + R_{2n}(x)$$

$\cos x \to f(x)$

$R_{2n}(x) \to$ Remainder Term

$$f(x) - P_{2n}(x) = R_{2n}(x)$$

$f(x) = \sin x \qquad -\frac{\pi}{4} \leq x \leq \pi/4$

$$|f(x) - P_n(x)| \leq 10^{-5}$$

$$\vdots$$

$$\left(\frac{\pi}{4}\right)^{n+1} \Big/ (n+1)!$$

find n so large that

$$(\pi/4)^{n+1} / (n+1)! \leq 10^{-5}$$

$\boxed{n=9}$ $\quad \frac{(\pi/4)^{10}}{10!} \approx 2.5e-8 < 10^{-5}$

$\boxed{n=5}$ $\quad \frac{(\pi/4)^{6}}{6!} \approx 0.000325 > 10^{-5}$

$\boxed{n=7}$ $\quad (\pi/4)^8 / 8! \approx 3.6e-06$

$\boxed{n=4}$ $\quad \frac{(\pi/4)^5}{5!} \simeq 0.0025 = 2.5e{-}03$

$\boxed{n=6}$ $\quad \frac{(\pi/4)^7}{7!} \simeq 0.000036 < 10^{-5}$

n should be at least 6 so that when we calculate $p_n(x)$

$$|\sin x - p_n(x)| < 10^{-5} \qquad -\pi/4 \leq x \leq \pi/4$$

<u>Remark</u> $\quad \frac{(\pi/4)^{n+1}}{(n+1)!} \overset{?}{<} 10^{-5}$

$$\log\left(\frac{(\pi/4)^{n+1}}{(n+1)!}\right) < \log(10^{-5})$$

Rule I $\quad \log(a^b) = b \log a$

Rule II $\quad \log\left(\frac{c}{D}\right) = \log c - \log D$

$$\log\left(\underline{(\pi/4)}^{n+1}\right) - \log((n+1)!) < -5 \log 10$$

Rule I

$$(n+1)\log \pi/4 - \log((n+1)!) < -5 \log 10$$

Rule I

$$-\log((n+1) * n * \cdots * 1)$$

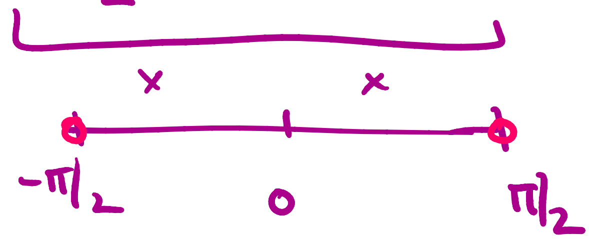
$2n \qquad n \to n+1 \qquad 2(n+1) \qquad 2n+2$

Lecture 02 Question

$$\left|\cos x - P_{2n}(x)\right| < 10^{-4}$$

formula $\longrightarrow$ $\left|R_{2n}(x)\right| = \left|\dfrac{x^{2(n+1)}}{(2(n+1))!} f^{(2(n+1))}(c_x)\right|$

$$\leq \frac{|x|^{2n+2}}{(2n+2)!} \left|f^{(2n+2)}(c_x)\right| \qquad -\pi/2 \leq x \leq \pi/2$$



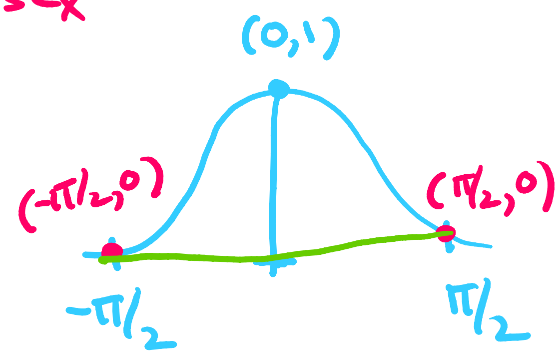
$x = \pi/2$

$$\leq \frac{(\pi/2)^{2n+2}}{(2n+2)!} \qquad \left|f^{(2n+2)}(c_x)\right|$$

$f(c_x) = \cos c_x$

$$= \frac{(\pi/2)^{2n+2}}{(2n+2)!} \qquad \left|(-1)^{n+1}\cos c_x\right|$$

$c_x = 0$



$$\leq \frac{(\pi/2)^{2n+2}}{(2n+2)!} \quad * \quad \cos 0$$

eliminated x & c_x from $|f(x) - P_{2n}(x)|$.

$$|f(x) - P_{2n}(x)| \leq \frac{(\pi/2)^{2n+2}}{(2n+2)!} \qquad -\pi/2 \leq x \leq \pi/2$$

pick n so that

$$(\pi/2)^{2n+2} \quad < 10^{-4}$$

pick n so that

$$\frac{(\pi/2)^{2n+2}}{(2n+2)!} < 10^{-4}$$

$n=3 \rightarrow 2n+2=8 \quad \frac{(\pi/2)^8}{8!} = 0.00091 = 9.1 * 10^{-4} > 10^{-4}$

$n=4 \rightarrow 2n+2=10 \quad \frac{(\pi/2)^{10}}{10!} = 0.000025 = 2.5 * 10^{-5} < 10^{-4}$

Conclusion: $2n \geq 8$ to achieve

$$|\cos x - p_{2n}(x)| < 10^{-4}$$

chap 2: any number x
2.125 can be written in the base "β" (Base means a positive integer)

β=2 → Binary format

$2.125 = 2^1 + 2^{-3}$

Worksheet 02: (Binary format → floating point representation)
#1

Express 45.12 using powers of 2 (Binary format)

$45 = 32 + 13$

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$2^0 \rightarrow 1$

$2^1 \rightarrow 2$

$\boxed{2^2 \rightarrow 4}$

$2^3 \rightarrow 8$

$2^4 \rightarrow 16$

$2^5 \rightarrow 32$

$2^6 \rightarrow 64$

$13 = 8 + 5$

$5 = 4 + 1 = 2^2 + 2^0$

$45 = 32 + 8 + 4 + 1 = 2^5 + 2^3 + 2^2 + 2^0$

$1 * 2^5 + 0 * 2^4 + 1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0$

1	0	1	1	0	1
2^5	2^4	2^3	2^2	2^1	2^0

This represents 45

Now focus on 0.12. Appendix E from Book.

Approach:

$0.12 * 2 = \boxed{0}.24$

$0.24 * 2 = \boxed{0}.48$

$0.48 * 2 = \boxed{0}.96$

$0.96 * 2 = \boxed{1}.92$

$0.92 * 2 = \boxed{1}.84$