Math 4329

Thursday, February 13, 2020 4:38 PM

$f(x) = x^6 - x - 1$

find an approximate x-intercept of $f(x)$ <u>same as</u>

<u>asking</u>

find the approximate root to

$f(x) = 0$.

or

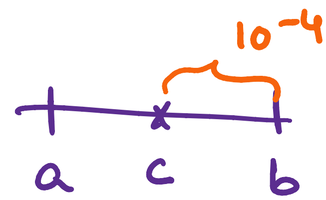
$-f(x) = 0$.

USING BISECTION METHOD with $\varepsilon = 10^{-4}$.

$\overset{-}{f(a)}\,\overset{+}{f(b)} < 0$, $a < b$.

$\overset{+}{}\quad\overset{-}{}$

B1 $\rightarrow$ $c = \frac{1}{2}(a+b)$



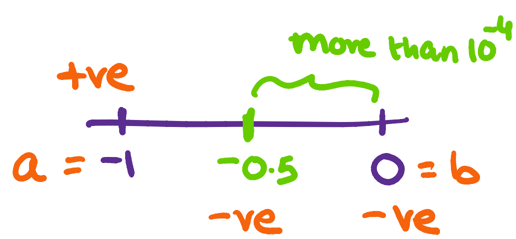
B2 $\rightarrow$ check if $b - c \le \varepsilon = 10^{-4}$.

B3 $\rightarrow$ check $f(c) * f(b) \le 0$

$f(x) = x^6 - x - 1$ $a = -1$, $b = 0$

$f(-1) = (-1)^6 - (-1) - 1 = 1 > 0$

$f(0) = -1 < 0$



ONE Iteration B1, B2, B3

B1 $\rightarrow$ $c = \frac{1}{2}(0 + -1) = -0.5$

B2 $\rightarrow$ $b - c = 0 - (-0.5) = 0.5 > 10^{-4} = \varepsilon$ go to B3

B3 $\rightarrow$ $f(c) = f(-0.5) < 0$

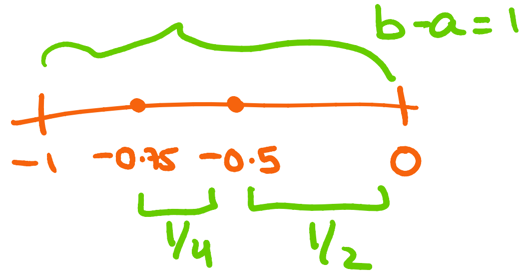
$f(b)\, f(c) > 0 \Rightarrow$ b & c are same sign

$\Rightarrow$ c is the new b

⇒ c is the new b

Replace b with c

$\boxed{a=-1 \qquad b=-0.5}$



Two Iteration B1-B3 . .

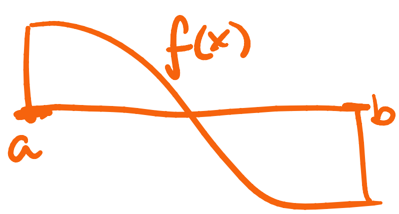
1(a) $xe^x = \cos x$

$f(x)=xe^x-\sin x$ $\quad f(0)=0$ excluded

$f(\pi) * f(-\pi/2) < 0$

equation $\boxed{xe^x = \cos x}$

$f(x)= xe^x - \cos x$ find the interval $[a,b]$ containing a root α.



$f(0)=0*e^0-\cos 0 = -1<0$

$f(\pi)=\pi e^\pi - \cos\pi = \pi e^\pi - (-1) = \pi e^\pi + 1 > 0.$

x measured in RADIANS NOT DEGREE.

$f(180^\circ) = 180^\circ * e^{180^\circ} - \cos 180^\circ$

$f(x)=xe^x-\cos x$

$\{a = 0, -\pi/2\}$ $\qquad$ $\{b = 1, \pi/2\}$

$$\left\{ \begin{array}{l} a = 0, -\pi/2 \\ a = 0 \end{array} \right\} \qquad \left\{ \begin{array}{l} b = 1, \pi/2 \\ b = \pi/6 \end{array} \right\}$$

$f(0) = -1$

$f(-\pi/2) = -\frac{\pi}{2} e^{-\pi/2} - 0 < 0$

$f(1) > 0$

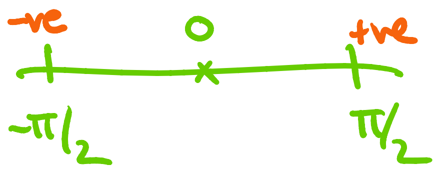
$f(\pi/2) = \pi/2\, e^{\pi/2} > 0$

$f(\pi/6) > 0$

$a = -\pi/2 \qquad b = \pi/2$

(b) $\quad a = -\pi/2 \qquad b = \pi/2$

$$[\underset{a}{-\pi/2}, \underset{b}{\pi/2}]$$



Iteration #1 of Bisection

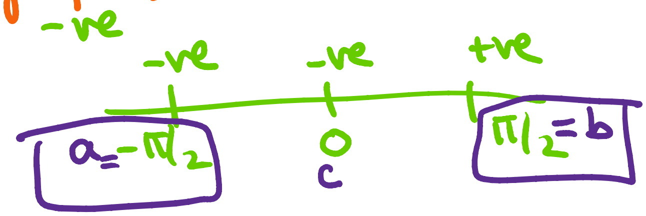
Start with $a = -\pi/2 \quad b = \pi/2 \quad \varepsilon = 10^{-4}$.

B1 $\rightarrow$ $c_1 = \frac{a+b}{2} = 0$

B2 $\rightarrow$ $b - c = \pi/2 > \varepsilon = 10^{-4}$ go to step B3.

B3 $\rightarrow$ $f(0) = 0e^0 - \cos 0 = -1 < 0$

sign $\underset{+ve}{f(b)}$ sign $\underset{-ve}{f(c)} < 0$



Replace a with c

$$\boxed{a = c = 0 \qquad \text{and} \qquad b = \pi/2}$$

Iteration #2

Iteration #2

$a = 0 \quad \frac{\pi}{4} \quad \pi/2 = b$

B1: $c_2 = \frac{0+\pi/2}{2}$

$= \pi/4$

B2: $b - c = \pi/2 - \pi/4 = \pi/4 > 10^{-4}$ go to B3.

B3: $f(c) = f(\pi/4) = \underbrace{\frac{\pi}{4}e^{\pi/4}}_{=1.01} - \cos\pi/4 \rightarrow 0.7226$

$\pi/4\, e^{\pi/4} = 1.7225981237$

0.7226

$f(c) > 0$

$-ve \quad +ve \quad +ve$

$a = 0 \quad \pi/4 \quad \pi/2 = b$

c

b gets replaced with c

$0e^0 - \cos 0$

-1

end of iteration #2

$a = 0 \quad b = \pi/4$

Iteration #3

$-ve \quad +ve$

$0 \quad \pi/8 \quad \pi/4$

B1: $c = \frac{1}{2}(a+b) = 1/2\,(0 + \pi/4) = \pi/8$

B2: If $b - c \leq \varepsilon$ then accept c as root

$\varepsilon \rightarrow$ Tolerance (my zero)

↳ Tolerance (my zero)

$b-c = \pi/8 > 10^{-4}$ go to B3.

B3: $f(c) = f(\pi/8) = \pi/8 \, e^{\pi/8} - \cos \pi/8 < 0$

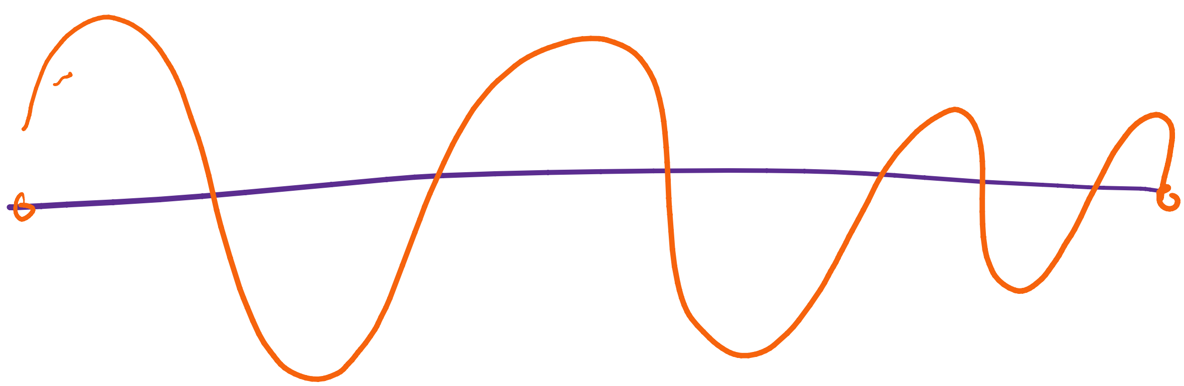
-ve -ve +ve

$a = 0$ $\frac{\pi}{8} = c$ $\pi/4 = b$

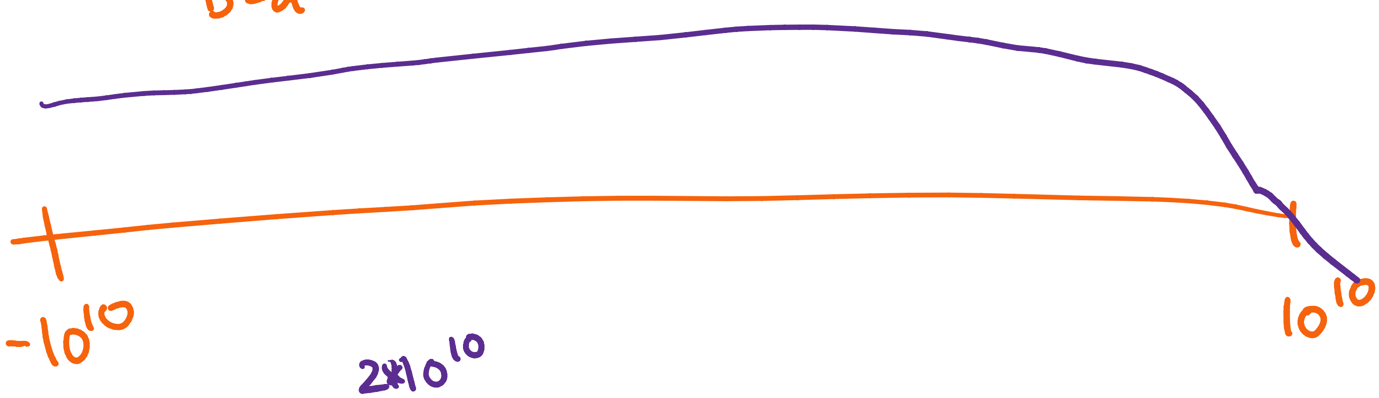
new a

$a = \pi/8$ $b = \pi/4$.

How many midpoints n need to be calculated so that I reach the root with $\varepsilon = 10^{-4}$?



b-a



$$|b-c| \leq \frac{b-a}{2^n} < \varepsilon$$

a b error between c & actual root

$\frac{b-a}{2^n}$.

4 (a) loss-of-significance errors.

$$f(x) = \frac{\sqrt{4+x} - 2}{x}$$

$x \to 0$ since $x = (0.01)^n$

$n = 1, \ldots 10$

$\sqrt{4+x} \to \sqrt{4+0} = 2$

x coming v.close to 0

so $\sqrt{4+x}$ is v.close to $\sqrt{4+0} = 2$

$\sqrt{4+x} - 2$ is very close to 0. This leads to a loss of significance when subtracting 2 numbers very close to each other.

$$f(x) = \frac{\sqrt{4+x} - 2}{x}$$

causes the loss of significance

$$= \frac{\sqrt{4+x} - 2}{x} * \frac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2}$$

$$= \left(\sqrt{4+x}\right)^2 - 2^2 \qquad = \cancel{4} + x - \cancel{4}$$

$$= \frac{\left(\sqrt{4+x}\right)^2 - 2^2}{x\left(\sqrt{4+x} + 2\right)} = \frac{\cancel{4} + x - \cancel{4}}{x\left(\sqrt{4+x} + 2\right)}$$

$$= \frac{x}{x\left(\sqrt{4+x} + 2\right)}$$

$$= \frac{1}{\sqrt{4+x} + 2}$$

mathematically equivalent function $g(x)$.

(b) $f(x) = \frac{x - \sin x}{x^3} * \frac{x + \sin x}{x + \sin x}$ does not help?

use Taylor represent^n of $\sin x$

$\sin x \approx p_3(x) = x - x^3/3!$

in $\frac{x - \sin x}{x^3} \approx \frac{x - p_3(x)}{x^3}$ approximately

$$= \frac{x - \left(x - x^3/3!\right)}{x^3} = \frac{0 + x^3/3!}{x^3}$$

$$= 1/3! = 1/6$$

L'HOPITAL Check: $\lim \frac{x - \sin x}{\ldots} \left(\frac{0}{\ldots}\right) \to 1 - \cos x \xrightarrow{0} \sin x \ldots$

L'Hopital

Check: $\lim_{x\to 0} \frac{x-\sin x}{x^3} \left(\frac{0}{0}\right) \to \frac{1-\cos x}{3x^2} \frac{0}{0} \to \frac{\sin x}{6x} \frac{0}{0} \to \frac{\cos x}{6} \to 1/6$

5 $\sqrt{1.5}$ $\qquad f(x) = \sqrt{x+1}$

$f(x) \approx p_3(x)$

$f(0.5) = \sqrt{1+0.5} = \sqrt{1.5}$

$p_3(0.5)$

$$p_3(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!}$$

$p_3(0.5)$

Ans: abs. error =

$|\text{True Value} - \text{approx. value}| = \boxed{0.2519}$ Ans

relative error $= \frac{\text{abs. error}}{|\text{True Value}|} = \frac{0.2519}{1.2247} \approx \boxed{0.20568}$ ANSWER

3 $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \frac{(-1)^n x^{2n+1}}{(2n+1)!} +$

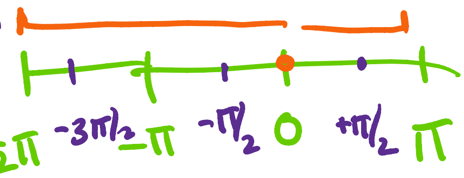
$\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \sin c$

$\sin x = p_{2n+1}(x) + $ Remainder Term.

$$\sin x = p_{2n+1}(x) + \text{Remainder Term.}$$

$$\left| \sin x - p_{2n+1}(x) \right| < 0.01 \qquad \boxed{-2\pi \le x \le \pi}$$

$$\left| \frac{x^{2n+3}}{(2n+3)!} \sin c \right| < 0.01$$



$$\frac{|x|^{2n+3}}{(2n+3)!} \left| \sin c \right| \le \qquad x = -2\pi \qquad c = \pi/2$$

$$\left| \sin x - p_{2n+1}(x) \right| \le \frac{|2\pi|^{2n+3}}{(2n+3)!} \left| \sin \pi/2 \right|$$

$$\le \frac{(2\pi)^{2n+3}}{(2n+3)!} * 1$$

$$< 0.01$$

Answer: $n = 9 \Rightarrow 2n+3 = 21$

& $\frac{(2\pi)^{21}}{21!} \simeq 0.001$

2 (a) $2^0 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{12} = 5113$

2(b) $e = 12 \quad E = 1023 + 12 = 1035 = 2^{10} + 2^3 + 2^1 + 2^0$

0	1	0	0	0	0	0	0	1	0	1	1	0	0	1	1	1	1	1	1	1	0	0	1

σ E   Decimal part of Mantissa.