Error in Poly Interpolation

Let $p_n^L(x)$ and $p_n^N(x)$ denote the Lagrange and Newton's D.D poly of degree n interpolating $n+1$ discrete & distinct data points:

$(x_0, y_0), \ldots, (x_n, y_n)$

<u>Note</u>: We have seen this poly. is unique poly. of degree n interpolating the $n+1$ data points. $(1,1), (4,2)$ & $(9,3) \to f(x) = \sqrt{x}$

Error in Interpolating: If $y_i = f(x_i)$ $i = 0, 1, \ldots n$.

Then,
$$f(x) - p_n^L(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!} f^{(n+1)}(c_x)$$

c_x unknown # bet. x & x_i

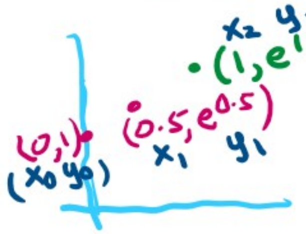
example: Without calculating $p_2^L(x)$, predict the "error in interpolating" $f(x) = e^x$ at $x = 0, 0.5, 1.$

min x-value, max x-value

$x = \underset{x_0}{0}, \underset{x_1}{0.5}, \underset{x_2}{1}$

$n = 2$ $\quad 0 \leq x \leq 1$

$$f(x) - p_2^L(x) = e^x - p_2^L(x)$$



$$\overset{n=2}{=} \frac{1}{3!}(x - \overset{x_0}{0})(x - \overset{x_1}{0.5})(x - \overset{x_2}{1}) f'''(c_x) \qquad 0 \leq c_x \leq 1$$

$$= \frac{1}{3*2*1} x(x-0.5)(x-1) e^{c_x} \qquad 0 \leq x \leq 1$$

$c_x = 1$

$$\leq \frac{1}{6}\left((x^2 - 0.5x)x - (x^2 - 0.5x)\right) e^1$$

$x^3 - 1.5x^2 + 0.5x$

$$= \frac{1}{6}\left(x^3 - 0.5x^2 - x^2 + 0.5x\right) e, \quad 0 \leq x \leq 1$$

$x = ?$ to give a maximum?

$g(x) = x^3 - 1.5x^2 + 0.5x$ maximum

$g(x) = x^3 - 1.5x^2 + 0.5x$ maximum

$g'(x) = 0$ verify that $g''(x) < 0$

$g'(x) = 0 \Rightarrow 3x^2 - 3x + 0.5 = 0$

solve for x value

$\underset{3}{a}x^2 + \underset{-3}{b}x + \underset{0.5}{c} = 0$

Roots $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $x_2 = \frac{-b - \sqrt{b^2-4ac}}{2a}$

$x_1 = \frac{-(-3) + \sqrt{9 - 4*3*0.5}}{6}$

$= \frac{3 + \sqrt{9 - 4*1.5}}{6} = \frac{3+\sqrt{3}}{6}$

$x_2 = (3 - \sqrt{3})/6$

assume maximum is at x_1

Recall: find

$f(x) - p_2^L(x) = \frac{1}{3!} \underbrace{x(x-0.5)(x-1)}_{x^3 - 1.5x^2 + 0.5x} e^{c_x}$ (Error formula)

$\leq 1/6 \quad g(x) \quad e^1 \qquad 0 \leq x \leq 1$

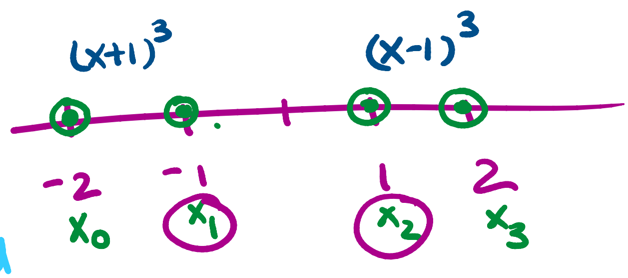
$\leq 1/6 \; g(x_1) \, e$ assumed that max attained at x_1

Spline Interpolation → Look at lecture slides.

$$S(x) = \begin{cases} (x+1)^3 & -2 \le x \le -1 \\ P_3(x) & -1 < x < 1 \\ (x-1)^3 & 1 \le x \le 2 \end{cases}$$

← spline $S(x)$

$S(x) = $ cubic poly on $[x_0, x_1]$, (x_1, x_2), $[x_2, x_3]$

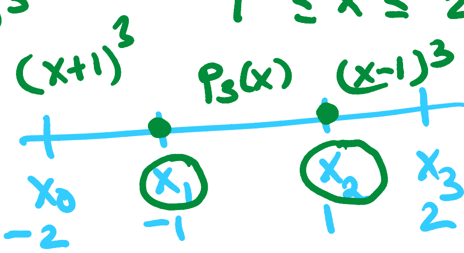


$P_3(x)$
$= ax^3 + bx^2 + cx + d$
(cubic polynomial)

Question Determine the cubic polynomial $P_3(x)$ so that

$$S(x) = \begin{cases} (x+1)^3 & -2 \le x \le -1 \\ P_3(x) & -1 < x < 1 \\ (x-1)^3 & 1 \le x \le 2 \end{cases}$$

is a cubic spline.



In other words,
find $P_3(x) = ax^3 + bx^2 + cx + d$:

Condn 1: $\displaystyle\lim_{x \to x_1^-} S'(x) = \lim_{x \to x_1^+} S'(x)$ holds

$\displaystyle\lim_{x \to x_1^-} S'(x) = \lim 3(x+1)^2 = 3(-1+1)^2 = 0$

and $\lim_{x \to x_1^-} s'(x) = \lim_{x \to -1} 3(x+1)^2 = 3(-1+1)^2 = 0$

$\lim_{x \to x_1^+} s'(x) = \lim_{x \to -1} P_3'(x) = \lim_{x \to -1} (3ax^2 + 2bx + c)$

$\lim_{x \to x_1^+} s'(x) = 3a + 2b(-1) + c$

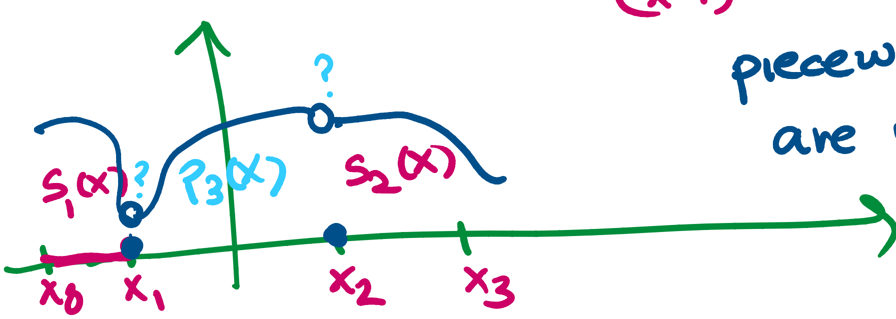
$= 3a - 2b + c$

at x_1, $\lim_{x \to x_1^-} s'(x) = \lim_{x \to x_1^+} s'(x)$

$3a - 2b + c = 0$

Goal: figure out a, b, c & d

Given $s(x) = \begin{cases} s_1(x) = (x+1)^3 & x_0 = -2 \le x \le -1 = x_1 \\ P_3(x) & -1 < x < 1 = x_2 \\ s_2(x) = (x-1)^3 & 1 \le x \le 2 = x_3 \end{cases}$



piecewise cubic poly. are meeting

$P_3(x) = ax^3 + bx^2 + cx + d$ (don't know a, b, c & d)

at $x = x_1 = -1$,

$s_1'(x_1) = P_3'(x_1)$
$s_1''(x_1) = P_3''(x_1)$ } smoothness of spline at x_1

$s_1''(x_1) = P_3''(x_1)$

$\hookrightarrow 0 = 3a - 2b + c$

we want at x_1, $\boxed{P_3''(x_1) = s_1''(x_1)}$

Recall: $P_3(x) = ax^3 + bx^2 + cx + d$

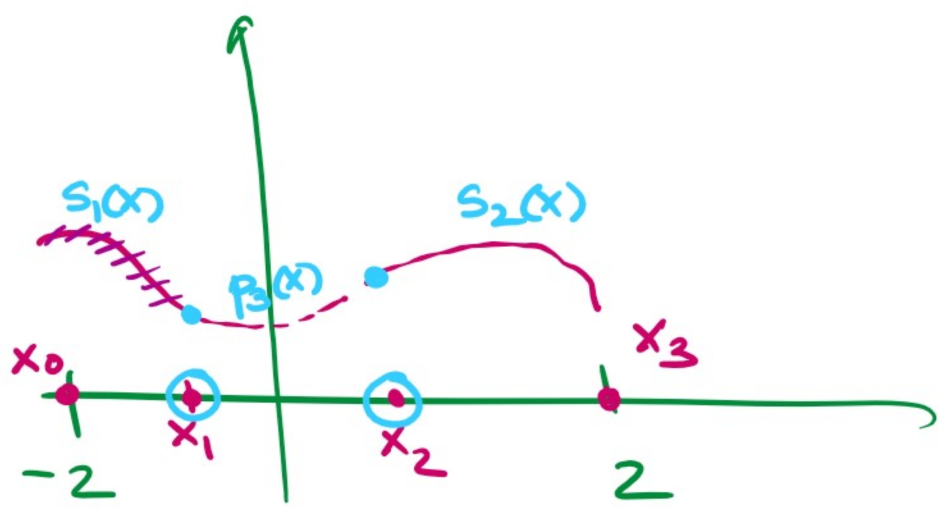
$P_3'(x) = 3ax^2 + 2bx + c$

$P_3''(x) = 6ax + 2b$

$P_3''(x_1) = s_1''(x_1)$, $x_1 = -1$

$-6a + 2b = 0$ $s_1(x) = (x+1)^3 \rightarrow s''(x) = 6(x+1)$

$s''(-1) = 6(-1+1) = 0$



Spline $S(x)$, $S'(x)$, $S''(x)$ should be Continuous on $[-2, 2]$.

s_1, P_3, s_2 polynomial

$S(x)$ $\lim_{x \to x_1^-} S(x) = \lim_{x \to x_1^+} S(x) =$ any same value

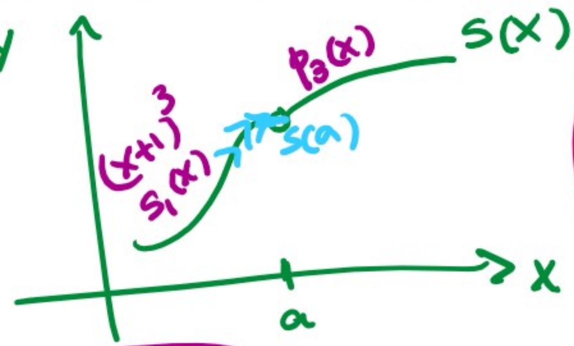
$\lim S'(x) = \lim S'(x) =$ any same value

$S(x)$ $\quad x \to x_1^- \qquad x \to x_1$

$$\lim_{x \to x_1^-} S'(x) = \lim_{x \to x_1^+} S'(x) = \text{any same value}$$

Afua's Question

Removable Discontinuity



$$\lim_{x \to a^-} S(x) \overset{?}{=} \lim_{x \to a^+} S(x)$$

LHL = RHL

$\neq S(a)$

$$\lim_{x \to a^-} S(x) = \lim_{x \to a^+} S(x) = S(a)$$

$$\left\{\begin{aligned} \lim_{x \to a^-} S(x) &= S(a) \\ \lim_{x \to a^-} (x+1)^3 &= S(a) \end{aligned}\right\} \quad \lim_{x \to a^-} S(x) = \underbrace{\lim_{x \to a^+} S(x)}_{= S(a)}$$

$$(a+1)^3 = S(a)$$

$$\lim_{x \to a^+} S(x) = S(a)$$

$$\lim_{x \to a^+} (\alpha x^3 + bx^2 + cx + d) = S(a)$$

$$\alpha a^3 + a^2 b + ac + d = S(a)$$

$$= (a+1)^3$$

find α, b, c & d:

$$\alpha a^3 + a^2 b + ac + d = (a+1)^3$$

$a=1$ $\alpha a^3 + a^2 b + ac + d = (a+1)^3$

$\alpha + b + c + d = 8$