Math 4329

Tuesday, April 28, 2020 4:36 PM

Quadrature Rules

Integration Rules

DoP $\quad \tilde{I}(f) = \underbrace{w_1}_{\text{weights}} f(x_1) + w_2 f(x_2) + \dots + w_k f(x_k)$ $\quad$ nodes

DoP $= 1$ if $\left.\begin{array}{l} f(x)=1 \\ f(x)=x^1 \end{array}\right\} \Rightarrow \tilde{I}(f) = \int_a^b f(x)\,dx = I(f)$

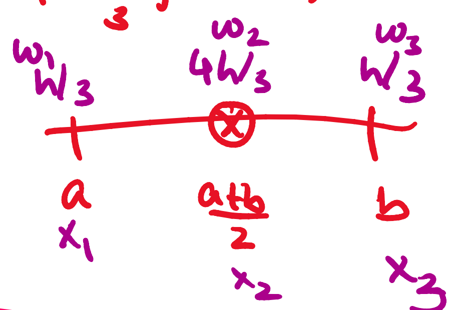
$f(x)=x^2 \Rightarrow \tilde{I}(f) \neq I(f)$

$h = \dfrac{b-a}{2}$

$w_1 = h/3 \quad x_1 = a \qquad w_2 = \dfrac{4h}{3} \quad x_2 = \dfrac{a+b}{2} \qquad w_3 = h/3, \ x_3 = b$

$$\tilde{I}(f) = h/3\, f(a) + \frac{4}{3} f\left(\frac{a+b}{2}\right) + \frac{h}{3} f(b), \quad h = \frac{b-a}{2}$$

$= S_2(f)$




<u>DoP 2</u>:

$\tilde{I}(f)$ $\quad \boxed{\frac{9}{4} f(-1) + \frac{3}{4} f(1) \approx \int_{-2}^{1} f(x)\,dx = I(f)}$

<u>claim</u>: $\tilde{I}(f)$ has DoP $= 2$ $\qquad \left.\begin{array}{l} f(x)=1 \\ f(x)=x \\ f(x)=x^2 \end{array}\right\} \Rightarrow \tilde{I}(f) = I(f)$

BUT $f(x) = x^3 \Rightarrow \tilde{I}(f) \neq I(f)$.

checking:

$f(x)=1 \Rightarrow \left.\begin{array}{l} \tilde{I}(f) = 9/4 + 3/4 = \frac{12}{4} = 3 \\ I(f) = \int_{-2}^{1} 1\,dx = 1-(-2) = 3 \end{array}\right\}$ ✓

$f(x)=x \Rightarrow \left.\begin{array}{l} \tilde{I}(f) = -9/4 + 3/4 = -6/4 = -3/2 \\ I(f) = \int_{-2}^{1} x\,dx = \frac{1}{2}(1-4) = -3/2 \end{array}\right\}$ ✓

$$I(f) = \int_{-2} x\,dx \quad = \frac{1}{2}(1-4) = -3/2$$

$$f(x) = x^2 \Rightarrow \hat{I}(f) = 9/4(-1)^2 + 3/4 = \frac{12}{4} = 3$$

$$I(f) = \int_{-2}^{1} x^2 dx \quad = \frac{1}{3}\left(1-(-8)\right) = 3 \quad \Big\} \checkmark$$

$$f(x) = x^3 \Rightarrow \hat{I}(f) = 9/4(-1)^3 + 3/4 = -6/4 = -3/2$$

$$I(f) = \int_{-2}^{1} x^3 dx \quad = 1/4\left(1-2^4\right) = -\frac{15}{4} = -5/3 \quad \Big\} \times$$

$$\Rightarrow \hat{I}(f) \neq I(f)$$

DoP $= 2$

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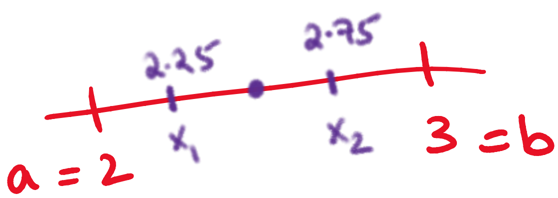
$$\int_{a=2}^{b=3} f(x)\,dx \quad \approx w_1 f(x_1) + w_2 f(x_2)$$




$$I(f) = \int_2^3 f(x)\,dx \quad \approx w_1 f(2.25) + w_2 f(2.75) = \hat{I}(f)$$

choose $w_1$ & $w_2$ so that the above integratn formula $\hat{I}(f)$ is exact for poly. of degree as large as possible. i.e, $\hat{I}(f)$ has DoP as high as possible.

(brute force!)

$f(x) = 1 \rightarrow \hat{I}(f) = I(f)$

$f(x) = x - \hat{I}(f) = I(f)$ (Brute force!)

$$w_1 f(2.25) + w_2 f(2.75) = \int_2^3 1\,dx = 3-2 = 1$$

$$\boxed{w_1 + w_2 = 1} \rightarrow (1)$$

$f(x) = x$

$$2.25 w_1 + 2.75 w_2 = \int_2^3 x\,dx = \frac{1}{2} x^2 \Big|_{x=2}^{3} = \frac{9-4}{2} = \frac{5}{2}$$

$f(x)=x \rightarrow \quad 2.25w_1 + 2.75w_2 = \int_2^3 x\,dx = \frac{1}{2} \left. x^2 \right|_{x=2}^{3} = \frac{9-4}{2} = \frac{5}{2}$

$$\boxed{2.25w_1 + 2.75w_2 = 2.5} \rightarrow 2$$

way to solve $w_1$ & $w_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2.25 & 2.75 & 2.5 \end{array}\right]$$

Augmented matrix approach

multiply 1 with $-2.25$ and add to equation 2

$$\begin{array}{rrcl} & -2.25w_1 - 2.25w_2 & = & -2.25 \\ + & 2.25w_1 + 2.75w_2 & = & 2.5 \\ \hline & 0 + 0.5w_2 & = & 0.25 \end{array}$$

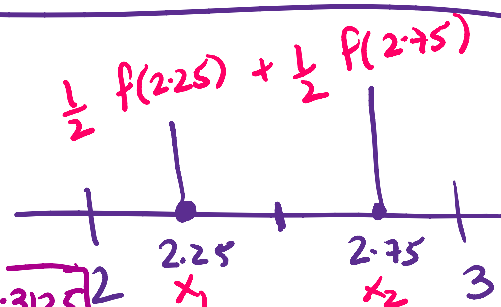
$$\boxed{w_2 = 0.5}$$

$w_1 + w_2 = 1 \rightarrow \boxed{w_1 = 0.5}$

formula becomes: $\boxed{\underbrace{\int_2^3 f(x)\,dx}_{I(f)} \approx \underbrace{0.5 f(2.25) + 0.5 f(2.75)}_{\tilde{I}(f)}}$

DoP = ?

$\left.\begin{array}{l} f(x) = 1 \\ f(x) = x \end{array}\right\} \Rightarrow \tilde{I}(f) = I(f)$




$f(x) = x^2$

$\tilde{I}(f) = \frac{1}{2}(2.25)^2 + \frac{1}{2}(2.75)^2 = 6.3125$

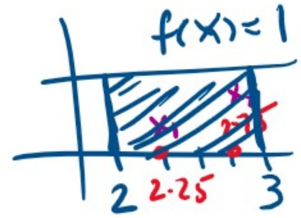
$I(f) = \int_2^3 x^2\,dx = \frac{1}{3} \left. x^3 \right|_{x=2}^{3} = \frac{1}{3}(27-8) = \frac{19}{3}$

$\tilde{I}(f) = 6.3125 \neq 6.333 = I(f)$

$\Rightarrow$ DoP $= 1$

Paraphrasing

$$I(f) = \int_2^3 f(x)\,dx \approx w_1 f(2.25) + w_2 f(2.75) = \check{I}(f)$$

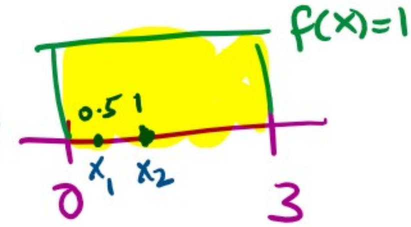



$$f(x)=1 \rightarrow \check{I}(f) = w_1 f(2.25) + w_2 f(2.75)$$
$$= w_1 + w_2$$
$$I(f) = \int_2^3 1\,dx = x\Big|_2^3 = 1$$

$$\check{I}(f) = I(f) \quad \text{when } f(x)=1$$
$$w_1 + w_2 = 1$$

example 2: $\displaystyle\int_0^3 f(x)\,dx \approx w_1 f(0.5) + w_2 f(1) = \check{I}(f)$

find $w_1$ & $w_2$ so that DoP of $\check{I}(f)$ is as high as possible.



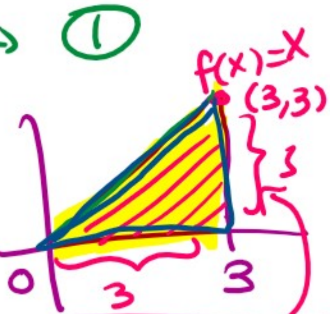

$$f(x)=1 \rightarrow \check{I}(f) = I(f)$$
$$\check{I}(f) = w_1 f(0.5) + w_2 f(1) = w_1 * 1 + w_2 * 1$$
$$= w_1 + w_2$$
$$I(f) = \int_0^3 1\,dx = 3$$

$$\boxed{w_1 + w_2 = 3} \longrightarrow ①$$

$$f(x)=x \rightarrow \hat{I}(f) = I(f)$$

$$w_1 f(0.5) + w_2 f(1)$$

$$\int_0^3 x\,dx = \frac{x^2}{2} = 9/2$$




$$\frac{1}{2}\text{Base} * \text{height} = 9/2 = 4.5$$

$$\boxed{0.5 w_1 + w_2 * 1 = 4.5} \quad ②$$

$$w_1 + w_2 = 3 \rightarrow ①$$
$$0.5 w_1 + w_2 = 4.5 \rightarrow ②$$

... with $-0.5$ and add to ②

$0.5w_1 + w_2 = 4.5 \rightarrow ②$

multiply ① with $-0.5$ and add to ②

$$\begin{array}{rrcl} & -0.5w_1 - 0.5w_2 & = & -1.5 \\ + & 0.5w_1 + w_2 & = & 4.5 \\ \hline & 0 + 0.5w_2 & = & 3 \rightarrow \boxed{w_2 = 6} \end{array}$$

① $\rightarrow$ $w_1 + w_2 = 3$ and $w_2 = 6$ gives

$w_1 + 6 = 3 \longrightarrow \boxed{w_1 = -3}$

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Least number of weights & nodes $w_1 f(x_1) + w_2 f(x_2)$ to get the highest DoP.

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2)$$

Can I find $w_1, x_1, w_2, x_2$ so the DoP is 3.

$\rightarrow$ 2 POINT Gaussian Quadrature (Book worked out $w_1, x_1$ $w_2, x_2$)

$w_1 = w_2 = 1$ and $x_1 = -\sqrt{3}/3$ $x_2 = \sqrt{3}/3$.

$$\int_a^b f(x)\,dx \rightarrow \int_{-1}^{1} f(u)\,k\,du$$ u-substitution

Numerical Analysis $\frac{d}{dx} f(x) \approx \frac{d}{dx} p(x)$ $\int_a^b f(x) \approx \int_a^b p(x)\,dx$

next lecture:

$$\begin{bmatrix} a & b \\ \end{bmatrix} \begin{bmatrix} x \\ \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \longrightarrow$$

* Newton's method

New method to

Next lecture: # Newton's method

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \longrightarrow$$ New method to solve $Ax = b$.

homework 10a #2

$\hat{I}(f)$

$$\int_0^{12} f(x)\,dx \approx \frac{\cancel{12}^{\,2}}{\cancel{6}}\left(f(0) + 4f(6) + f(12)\right)$$

$$2\left(f(0) + 4f(6) + f(12)\right)$$

$f(x) = x$

$$2(24 + 12)$$

$$36$$

$$I(f) = \int_0^{12} x\,dx = \frac{144}{2} = 72$$

$$f(x) = x^2 \quad I(f) = \frac{x^3}{3}\Big|_0^{12} = \frac{12^3}{3} = \underline{\qquad}$$

$$\hat{I}(f) = 2\left(f(0) + 4f(6) + f(12)\right)$$

$$4*36 + 144$$