Math-4329

Tuesday, May 5, 2020 4:24 PM

Solve
$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 6x_2 + 8x_3 &= 3 \\ 6x_1 + 8x_2 + 18x_3 &= 5 \end{aligned}$$

using LU decomposition & forward / Backward substitution.

$$A = \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

a_{21}, a_{31}

$R_2 \to R_2 - m_{21}R_1$ $\quad m_{21} = 2/1 = 2$

$R_3 \to R_3 - m_{31}R_1$ $\quad m_{31} = 6$

$$\begin{array}{c} \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 2 & 12 \end{bmatrix}$$

a_{32}

$m_{32} = \frac{a_{32}}{a_{22}} = 1/2$ $\quad R_3 \to R_3 - m_{32}R_2$

$R_3 \to R_3 - 1/2\, R_2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

$$L \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 1/2 & 1 \end{bmatrix}$$

Forward sub. $Ly = b$

Backward sub. $Ux = y$

Answer

$y_1 = 1 \quad y_2 = 1 \quad y_3 = -1.5$

$x_1 = 2/3, \quad x_2 = 1/2 \quad x_3 = -1/6$

$Ax = b$

LU-Decomposition

WS 10b) $\to$ System III

$$\begin{aligned} x + y &= 0 \\ x + \tfrac{401}{400} y &= 20 \end{aligned}$$

$\frac{401}{400} = 1.0025$ Suppose we want to solve this system III on a computer which only permits 2 digits after the decimal.

$\frac{401}{400} = 1.0025 \approx 1.00$

$$\left.\begin{aligned} x+y &= 0 \\ x + \tfrac{401}{400}y &= 20 \end{aligned}\right\} \to \begin{aligned} x+y &= 0 \\ x + y &= 20 \end{aligned}$$

NO SOLUTION

$\frac{401}{400} \to$ multiply $x + \frac{401}{400}y = 20$ by 400

$\hookrightarrow 400x + 401y = 20 * 400$

8000 $\quad$ -8000

Conditioning of Linear System: $Ax = b$

WS 10b #2
$$\left\{\begin{aligned} x + y &= 2 \\ x + 1.0001y &= 2 + \alpha, \end{aligned}\right. \quad \begin{aligned} \alpha &= 0 \\ \alpha &= 10^{-3} \\ \alpha &= 10^{-4} \end{aligned}$$

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$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 2+\alpha \end{bmatrix}$$

Relatively Small change in $b = \begin{bmatrix} 2 \\ 2+\alpha \end{bmatrix}$ $\quad \alpha = 0, \ \alpha = 10^{-3} \quad \alpha = 10^{-4}$

Results in a relatively large change in x.

$\alpha = 0 \quad x = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \alpha = 10^{-3} \quad x = \begin{bmatrix} 0.99999\ldots632 \\ 1.00\ldots0368 \end{bmatrix} \quad \alpha = 10^{-4} \quad x = \begin{bmatrix} 1.8999\ldots9817 \\ 0.1000\ldots021 \end{bmatrix}$

small change in RHS $\vec{b} \implies$ large change in x

Norm of vector: magnitude of vector

Norm of matrix:

Small change in RHS $\vec{b}$ ⟹ large change in x

Norm of vector: magnitude of vector

Norm of matrix:

magnitude

norm of $a_1 = -2$ $\quad |-2| = 2 \quad |a_1| = 2$

norm of $\vec{a} = \begin{pmatrix} -2 \\ -1000 \end{pmatrix}$ $\quad \|\vec{a}\| = \max\{|-2|, |-1000|\}$ $= 1000$

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Norm of A =

$$A = \begin{bmatrix} -2 & -100 \\ 5 & 10^5 \end{bmatrix} \begin{array}{l} \text{Row Sum} \\ \to |-2| + |-100| = 102 \\ \to 10^5 + 5 = 10^5 + 5 \end{array}$$

take max 10^5+5 = matrix norm of A

Row Sum of magnitude of each row entry

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{array}{l} 1+0 = 1 \\ 2+1 = 3 \end{array}$$

$$\boxed{\|A\| = 3}$$

WS 10b #2 $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$ is ill-conditioned

because small changes in $\vec{b} = \begin{pmatrix} 2 \\ 2+\alpha \end{pmatrix}$ lead to disproportionately large changes in $\vec{x}$.

$Ax = b$

Condition Number of $A = \|A\| * \|A^{-1}\|$

formula $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad \det A = ad - bc$

$A * A^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1} * A$

Property of $\|AB\| \leq \|A\| \, \|B\|$ (Property P3, Page 298)

$B = A^{-1}$ $\quad \|AA^{-1}\| \leq \|A\| \, \|A^{-1}\|$

$\|I\| \leq \|A\| \, \|A^{-1}\|$

$\|I\| = \left\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\| = 1$

$1 \leq \|A\| \, \|A^{-1}\| = \text{condn}(A)$

When condn(A) is very large the solution to $Ax = b$ will be very sensitive relatively small changes in b.

Such systems are called ill conditioned

WS 10b / #2 $\quad x + y = 2$

$x + 1.0001y = 2 + \alpha$, $\alpha = 10^{-3}, 10^{-4}$

$\vec{b}$ input

solution

$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\begin{bmatrix} -8 \\ 10 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2.001 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}$

A is ill conditioned!

$\|A\| * \|A^{-1}\| = \text{cond}(A)$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{array}{l} \text{Row Sum} \\ 1+1 = 2 \\ 1+1.0001 = 2.0001 \end{array}$$

$2.0001 \to$ max

$$A=\begin{bmatrix}1 & 1\\ 1 & 1.0001\end{bmatrix} \quad \begin{matrix}1+1 & = 2\\ 1+1.0001 & = 2.0001 \rightarrow \text{max}\end{matrix}$$

$\|A\| = 2.0001$

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \qquad A^{-1} = \frac{1}{\det A}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

$\det A = ad - bc$

$$A=\begin{bmatrix}1 & 1\\ 1 & 1.0001\end{bmatrix} \quad \det A = 1.0001 - 1 = 0.0001 = 10^{-4}$$

$$A^{-1} = \frac{1}{10^{-4}}\begin{bmatrix}1.0001 & -1\\ -1 & 1\end{bmatrix} \rightarrow A^{-1} = 10^{4}\begin{bmatrix}1.0001 & -1\\ -1 & 1\end{bmatrix} \begin{matrix}\text{Row Sum}\\ 2.0001\\ 2\end{matrix}$$

$\|A^{-1}\| = 10^4 * 2.0001$

$\|A^{-1}\| = 20{,}001$

$$\text{cond}(A) = \|A\| * \|A^{-1}\| = 2.0001 * 20{,}001 = 40{,}004.0001$$

very large number

$\Rightarrow$ A is ill conditioned & so sensitive to small changes in b leading to large changes in x

$Ax = b \quad x = \begin{bmatrix}2\\0\end{bmatrix} \quad \alpha = 0 \quad b = \begin{pmatrix}2\\2\end{pmatrix}$

$A\hat{x} = \hat{b} \quad \hat{b} = \begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}0\\10^{-3}\end{bmatrix} \quad \alpha = 10^{-3}$

$\hat{x} = \begin{pmatrix}-8\\10\end{pmatrix}$

$$\text{Rel}(x) = \frac{\|x-\hat{x}\|}{\|x\|} \leq \underbrace{\|A\|\,\|A^{-1}\|}_{\text{Cond}(A)} \; \frac{\|b-\hat{b}\|}{\|b\|} \rightarrow \text{Rel}(b)$$

Page 299, (6.89)

$x = \begin{pmatrix}2\\0\end{pmatrix} \quad \hat{x} = \begin{pmatrix}-8\\10\end{pmatrix} \qquad x - \hat{x} = \begin{pmatrix}2-(-8)\\0-10\end{pmatrix} = \begin{pmatrix}10\\-10\end{pmatrix}$

$\|x\| = 2 \qquad \|x-\hat{x}\| = 10, \quad \|x\| = 2$

$$\text{Rel}(x) = \frac{10}{2} = 5$$

$$\text{Rel}(b) = \frac{\|b-\hat{b}\|}{\|b\|} \qquad b = \begin{pmatrix}2\\2\end{pmatrix} \quad \hat{b} = \begin{pmatrix}2\\2+10^{-3}\end{pmatrix}$$

$b - \hat{b} = \begin{pmatrix}0\\-10^{-3}\end{pmatrix} \quad \|b-\hat{b}\| = 10^{-3}$

$\|b\| = 2$

$= \frac{10^{-3}}{2}$

$= 0.5 * 10^{-3} = 0.0005$

Rel(b) 0.0005

$= 0.5 * 10^{-3} = 0.0005$

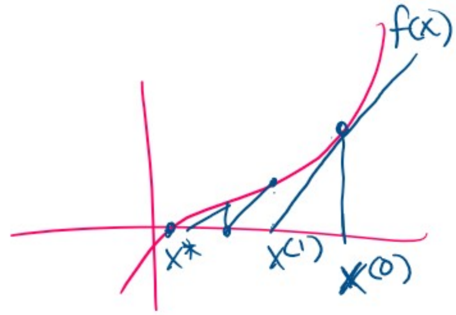
$5 = Rel(X) \leq \underbrace{\|A\| * \|A^{-1}\|}_{40{,}004.0001} * Rel(b) \rightarrow 0.0005$

Iterative methods : $X^{(0)} \rightarrow X^{(1)}, X^{(2)}, X^{(3)}, \ldots \rightarrow X^*$

Recall: $\underbrace{e^x - \cos x}_{f(x)} = 0$ using Newton's method.

$x^{(0)} = 5$

$x^{(1)} = x^{(0)} - \dfrac{f(x^{(0)})}{f'(x^{(0)})}$



Goal: Solve $A x = b$ using iterative method.

$X^{(0)} = \begin{pmatrix} x_{00} \\ y_{00} \end{pmatrix}$ formula $X^{(1)}$

SysI / WS 10b

$10x_1 - x_2 = 1$

$2x_1 + 5x_2 = 2$

$X^{(0)} = \begin{pmatrix} 0 \; x_1^{(0)} \\ 0 \; x_2^{(0)} \end{pmatrix}$ $X^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}$?

$x_1^{(1)} \leftarrow 10x_1^{(1)} - x_2^{(0)} = 1$ equation 1

$x_2^{(1)} \leftarrow 2x_1^{(0)} + 5x_2^{(1)} = 2$ equation 2

$x_1^{(0)} = x_2^{(0)} = 0$

$10x_1^{(1)} - 0 = 1 \rightarrow x_1^{(1)} = 1/10$

$2*0 + 5x_2^{(1)} = 2 \rightarrow x_2^{(1)} = 2/5$

$\boxed{x_1^{(1)} = 1/10}$ $\quad \boxed{x_2^{(1)} = 2/5}$

$x_1^{(2)} \leftarrow 10x_1^{(2)} - \boxed{x_2^{(1)}} = 1$ equation 1

$x_2^{(2)} \leftarrow 2\boxed{x_1^{(1)}} + 5x_2^{(2)} = 2$ equation 2

eqn ① → $10x_1^{(2)} - 2/5 = 1 \rightarrow x_1^{(2)} = 7/50$

eqn ② → $2 * \frac{1}{10} + 5x_2^{(2)} = 2 \rightarrow x_2^{(2)} = \frac{1.8}{5}$

$1/5$