CPS-5310

Thursday, April 16, 2020 1:29 PM

Homework 4

#1 Consider the latest data published from El Paso county.

* lsqcurvefit } → needs exact solution
* fminsearch
* Numerical Derivative approach

→ single-species general population model: #6, homework 3

#2 Harmonic Oscillator → long term behavior

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + f(t,x) = 0$$ (2nd order ODE)

$$\frac{d\vec{Y}}{dt} = \vec{F}(t,\vec{Y})$$

eqbm points/displacement & determine the long-term behavior.

Last Lecture: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$

$k>0$ — $k$, $0$, $-k$ Phase Lines to determine long term behavior

Phase Portraits → generalized 2D analogue of phase lines.

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \qquad \vec{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$y_1 = x$ ; $y_2 = \frac{dx}{dt} = \frac{dy_1}{dt}$

2 first-order ODEs:

$$y_1 = x \quad ; \quad y_2 = \frac{dx}{dt} = \dot{x}$$

2nd order ODE $\rightarrow$ sys of first order ODEs:

$$dy_1/dt = y_2 \rightarrow f_1$$

$$m\, dy_2/dt + b y_2 + k y_1 = 0 \Rightarrow \frac{dy_2}{dt} = \underbrace{-\frac{1}{m}(k y_1 + b y_2)}_{f_2}$$

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \qquad \frac{d\vec{Y}}{dt} = \begin{bmatrix} dy_1/dt \\ dy_2/dt \end{bmatrix}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \Rightarrow \frac{d\vec{Y}}{dt} = A\vec{Y}$$

$$A\vec{Y} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} y_2 \\ -\frac{k}{m} y_1 - \frac{b}{m} y_2 \end{bmatrix} \end{array} \begin{array}{l} f_1 \\ \rightarrow f_2 \end{array}$$

Long Term behavior of $\frac{d\vec{Y}}{dt} = A\vec{Y}$ ?

Equilibrium Pts of the vector-valued ODE

$$\frac{d\vec{Y}}{dt} = \vec{0} \Rightarrow A\vec{Y} = \vec{0}$$

Assume: A is nonsingular $\det A \neq 0$

$\Rightarrow$ $\vec{Y} = \vec{0}$ is the only soln.

Goal: What kind of eqbm pt is $\vec{0}$ ?


$$\frac{dY}{dt} = AY, \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$$

$\vec{Y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is eqbm point.

Phase plane (generalizatn of line) detemnes the behavior

Online harmonic oscillator problem: $m=1$ $b=7$, $k=10$.

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}$$

Long Term behavior, solutions to $\frac{d\vec{Y}}{dt} = A\vec{Y}$ :

$$\vec{Y} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

where $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$ are eigen valve eigen vector pairs to $A$.

i.e $\vec{v}_i \neq \vec{0}$ s.t. $A\vec{v}_i = \lambda_i \vec{v}_i$ $i = 1, 2$.

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}$$

determine $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$

$$\vec{Y} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

find e.values $\lambda_1$ & $\lambda_2$ and then e.vectors $\vec{v}_1$ & $\vec{v}_2$.

... to characteristic eq$^n$:

find e.values $\lambda_1$ & $\lambda_2$ -

e.values solutions $\lambda$ to characteristic eqn:

determinant $(A - \lambda \underbrace{I}_{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}) = 0$

$$A - \lambda I = \begin{bmatrix} 0-\lambda & 1 \\ -10 & -7-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow (-\lambda)(-7-\lambda) - (-10) = 0$$

$$\lambda^2 + 7\lambda + 10 = 0$$

$$\Rightarrow \underbrace{\lambda = -2}_{\lambda_1} \text{ or } \underbrace{\lambda = -5}_{\lambda_2}$$

$\lambda_1 = -2$ calculate $\vec{v}_1$ ?

$$\vec{v}_1 = \begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix} \to \quad A\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$\begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} -2y_{11} \\ -2y_{21} \end{bmatrix}$$

$$\boxed{y_{21} = -2y_{11}}$$

$$-10y_{11} - 7y_{21} = -2y_{21}$$

$y_{11} \to$ free variable

$y_{21} = -2y_{11}$

$y_{11} = 1 \to y_{21} = -2$

$y_{11} = 1/2 \quad \begin{pmatrix} 1/2 \\ -1 \end{pmatrix} = \vec{v}_1$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\boxed{\lambda_1 = -2, \ \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

$\boxed{\text{Similarly, } \lambda_2 = -5 \ \ \vec{v}_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}}$ (Please confirm!)

$$\vec{x} = c_1 e^{-2t}\begin{pmatrix} 1 \\ \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ \end{pmatrix}$$

General soln: $\vec{Y} = c_1 e^{-2t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2 e^{-5t}\begin{pmatrix}1\\-5\end{pmatrix}$

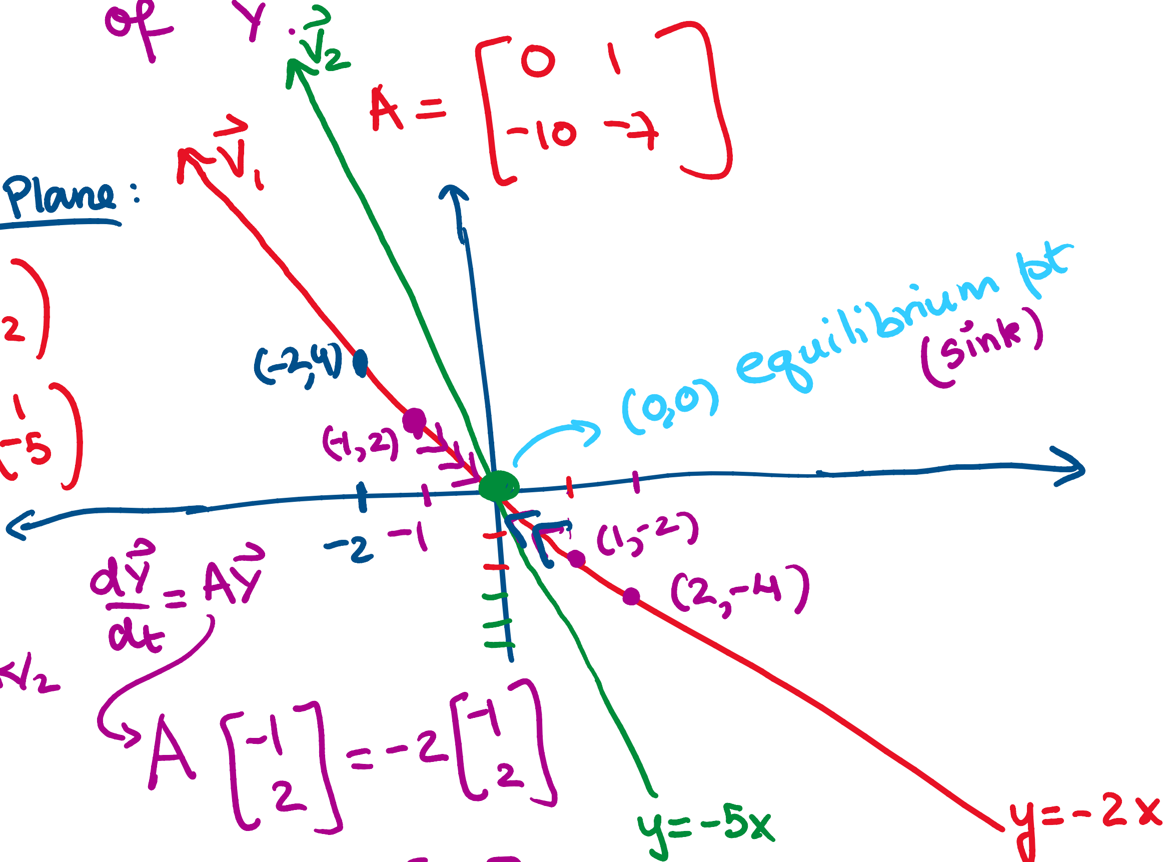
where $c_1$ & $c_2$ are consts determined by initial value of $\vec{Y}$.

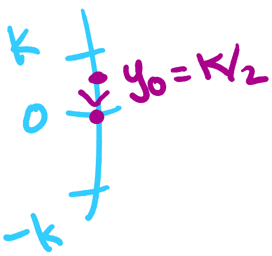
$A = \begin{bmatrix}0 & 1\\-10 & -7\end{bmatrix}$

Phase Plane:

$\vec{v}_1 = \begin{pmatrix}1\\-2\end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix}1\\-5\end{pmatrix}$




$\frac{d\vec{Y}}{dt} = A\vec{Y}$

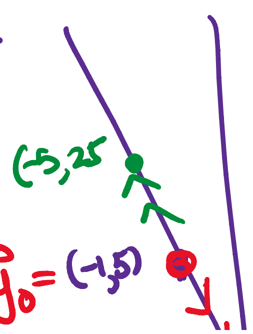
$A\begin{bmatrix}-1\\2\end{bmatrix} = -2\begin{bmatrix}-1\\2\end{bmatrix}$

$A\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}2\\-4\end{bmatrix}$

$A\begin{bmatrix}1\\-2\end{bmatrix} = -2\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}-2\\4\end{bmatrix}$

$\begin{bmatrix}1\\-5\end{bmatrix} = \vec{v}_2$

$A = \begin{bmatrix}0 & 1\\-10 & -7\end{bmatrix}$



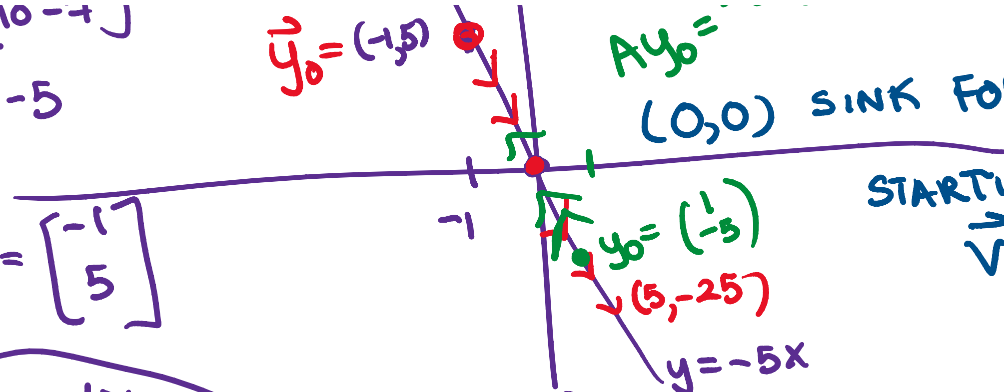

$A\vec{y}_0 = -5\begin{pmatrix}1\\-5\end{pmatrix} = \begin{pmatrix}-5\\25\end{pmatrix}$

SINK FOR SOLUTIONS

$\begin{bmatrix} 10 & -4 \end{bmatrix}$

$\lambda_2 = -5$

$\vec{y}_0 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$




(0,0) SINK FOR SOLUTIONS STARTING ON $y = -5x$

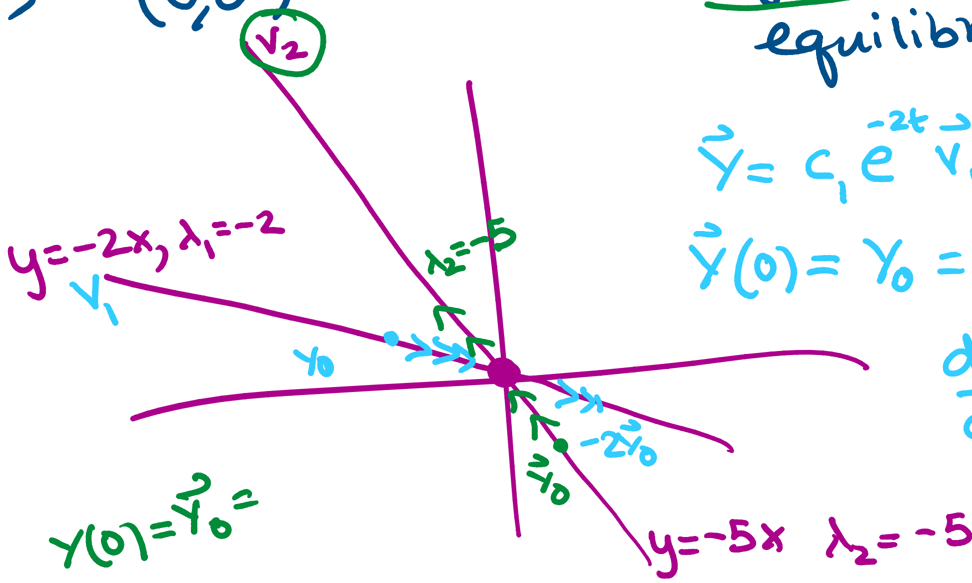
$\vec{V}_2$

$A\vec{y}_0 = \frac{dy}{dt} \rightarrow A\vec{y}_0 = -5\vec{y}_0$

$\vec{y}_0 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$A\vec{y}_0 = -5\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -25 \end{pmatrix}$

$\Rightarrow$ (0,0) is a sink asymptotically stable equilibrium point.




$\vec{y} = c_1 e^{-2t}\vec{v}_1 + c_2 e^{-5t}\vec{v}_2$

$\vec{y}(0) = y_0 = c_1 e^0 \vec{v}_1$

$\frac{d\vec{y}}{dt} = A\vec{y}$

$y(0) = \vec{y}_0 =$

$$\lim_{t \to \infty} \vec{y}(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$1/e^{2t} \rightarrow 1/\infty = 0$

$e^{-5t} \rightarrow 0$ as $t \to \infty$

$\vec{y}$ solves $\frac{d\vec{y}}{dt} = A\vec{y}$

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

$$\frac{d\vec{Y}}{dt} = c_1(-2e^{-2t})\begin{pmatrix}1\\-2\end{pmatrix} + c_2(-5e^{-5t})\begin{pmatrix}1\\-5\end{pmatrix}$$

$$= c_1(-2)e^{-2t}\begin{pmatrix}1\\-2\end{pmatrix} + (-5)c_2e^{-5t}\begin{pmatrix}1\\-5\end{pmatrix}$$

$\lambda_1$ $\quad\vec{v}_1$

$A\vec{v}_1$

$\lambda_2\vec{v}_2 = A\vec{v}_2$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + f(t,x) = 0$$

need not be $kx$

$\sin(\pi t)$

Determine long term behavior

$$\frac{d\vec{Y}}{dt} = \vec{F}(t,\vec{Y})$$