Math 3323, Final

Name `____________________________`

1. If A =

$$\begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

   a. Find all eigenvalues of A.

   answer: $\lambda = 2$

   b. For each eigenvalue, find a basis for the subspace of eigenvectors (the eigenspace).

   answer: for $\lambda = 2$, basis is $[(1, 0, 0), (0, -1, 1)]$.

   c. Is A defective?

   answer: yes

2. Find the eigenvalues of A =

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 8 & 1 \end{bmatrix}$$

answer: $3, 3, 1+4i, 1-4i$.

3. Find the general solution of:

$$\begin{aligned} \frac{dx}{dt} &= 4x + 2y \\ \frac{dy}{dt} &= x + 3y \end{aligned}$$

answer: $\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4. If A =

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 4 & 0 & 8 & -4 & 12 \\ -2 & 0 & -4 & 2 & -6 \end{bmatrix}$$

a. Find a basis for the range of A.

answer: $[(1, 4, -2)]$

b. Find a basis for the null space of A.

answer: $[(0, 1, 0, 0, 0), (-2, 0, 1, 0, 0), (1, 0, 0, 1, 0), (-3, 0, 0, 0, 1)]$

c. What is the rank and what is the nullity of A? What do the rank and nullity always add up to?

answer: rank=1, nullity=4. rank+nullity=n (number of columns)

d. What is the dimension of the row space?

answer: 1

5. Find the equation of the plane through the points $(3, 2, 3), (1, 1, 4), (6, 2, 5)$.

answer: $2x - 7y - 3z + 17 = 0$

6. Prove the triangle inequality, $\|x + y\| \leq \|x\| + \|y\|$. You can use the fact that $|x^T y| \leq \|x\|\|y\|$ (in R^2 and R^3 this follows from $x^T y = \|x\|\|y\| \cos(\theta)$). (Hint: write $\|x + y\|^2 = (x + y)^T (x + y)$ and expand.)

answer: $\|x + y\|^2 = (x + y)^T (x + y) = x^T x + 2x^T y + y^T y = \|x\|^2 + 2x^T y + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$
so $\|x + y\| \leq \|x\| + \|y\|$ (This was homework problem 25, section 3.6)

7. True or False:

   a. If a subspace W has a basis of 5 vectors, any set of 7 vectors in W must be linearly dependent. (true)

   b. If a subspace W has a basis of 5 vectors, any set of 7 vectors in W must span W. (false)

   c. Any set of vectors that includes the zero vector must be linearly dependent. (true)

   d. If a subspace W has a basis of 5 vectors, any set of 5 independent vectors in W must span W. (true)

   e. The algebraic multiplicity of an eigenvalue is always less than or equal to the geometric multiplicity. (false)

   f. The volume of a parallelepiped with adjacent edges u,v,w is $|u \bullet v \times w|$ (true)

   g. The vector x which minimizes $\|Ax - b\|$ is $x = (A^TA)^{-1}A^Tb$, if A^TA has an inverse. (true)

   h. $(AB)^T = A^TB^T$ (false)

   i. A^TA and AA^T are always symmetric. (true)

   j. The columns of a square matrix are independent if and only if the rows are independent. (true)