Bases and Dimension

spanning set: A set of vectors $[w_1, ..., w_m]$ is a spanning set for a subspace W if any vector w in W can be written as a linear combination
$w = a_1w_1 + ... + a_mw_m$.

linear independence: A set of vectors $[w_1, ..., w_m]$ is linearly independent if $a_1w_1 + ... + a_mw_m = 0$ implies $a_1 = ... = a_m = 0$.

basis: A basis for W is a linearly independent spanning set for W.

Theorem: If $[w_1, ..., w_p]$ is a basis for W, any vector w in W can be written in one and only one way as a linear combination $w = a_1w_1 + ... + a_pw_p$.

Proof: A basis is a spanning set, so any vector w in W can be written in at least one way as a linear combination $w = a_1w_1 + ... + a_pw_p$. To show there is only one way to do this, assume there is another way, $w = b_1w_1 + ... + b_pw_p$. Then, subtracting, we have $0 = (a_1 - b_1)w_1 + ... + (a_p - b_p)w_p$ and since the vectors $w_1, ..., w_p$ are independent, $a_1 - b_1 = ... = a_p - b_p = 0$ so $a_i = b_i$ for all i.

Theorem: All bases for a given subspace W have the same number of vectors. (So we can call this number the dimension of W.)

Proof: Assume there are two bases $[w_1, ..., w_p]$ and $[u_1, ..., u_q]$ with $q > p$ and we will show that the u_i cannot be linearly independent. Suppose $x_1u_1 + ...x_qu_q = 0$. Since the w_i are a spanning set, each u_i can be written as a linear combination of $w_1, ..., w_p$, so:

$x_1(a_{11}w_1 + ... + a_{p1}w_p) + ... + x_q(a_{1q}w_1 + ... + a_{pq}w_p) = 0$, or
$(x_1a_{11} + ... + x_qa_{1q})w_1 + ... + (x_1a_{p1} + ... + x_qa_{pq})w_p = 0$.

But since the w_i are linearly independent, the coefficients of each w_i must be zero:

$$\begin{aligned} x_1a_{11} + ... + x_qa_{1q} &= 0 \\ &. \\ &. \\ x_1a_{p1} + ... + x_qa_{pq} &= 0 \end{aligned}$$

This is a homogeneous linear system of p equations for the q unknowns $x_1, ..., x_q$, and since there are more unknowns than equations, there must be a nonzero solution. Thus, the u_i are linearly dependent and cannot form a basis. (Note that in the proof, we have shown that any set of more than p vectors in a p dimensional subspace must be linearly dependent.)

Theorem: If $p = dim(W)$, any set of p independent vectors is a basis for W.

Proof: If $[v_1, ..., v_p]$ is linearly independent, then to show it is a basis we just need to show it spans W, that is, that any w in W can be written as a linear combination of the v_i. As shown in the proof of the previous theorem, $[v_1, ..., v_p, w]$ must be dependent, because it consists of $p+1$ vectors in a p dimensional space. So $a_1v_1 + ... + a_pv_p + a_0w = 0$ for some set of numbers $a_0, a_1, ..., a_p$ which are not all 0. But a_0 must be one of the nonzero numbers, because if $a_0 = 0$, we have $a_1v_1 + ... + a_pv_p = 0$ for some set of numbers $a_1, ..., a_p$ which are not all 0, and that is impossible, because the v_i are independent. So we can solve for $w = -(a_1/a_0)v_1 - ... - (a_p/a_0)v_p$, which shows that an arbitrary w in W can be written as a linear combination of the $[v_1, ..., v_p]$, and so this set is a spanning set, and thus a basis for W.

If $[w_1, ..., w_m]$ is a set of vectors in a subspace W of dimension p, any two of the following guarantees the third:

1. The number of vectors $m = p$.
2. The set is linearly independent.
3. The set is a spanning set for W.

	$m < p$	$m = p*$	$m > p$
w_i are independent?	maybe	maybe	no
w_i form spanning set?	no	maybe	maybe
w_i are basis?	no	maybe	no

*All three yes or all three no.