$f(x) = 0$

$f(x) \approx T_1(x) = f(x_n) + f'(x_n)(x - x_n) = 0$

$x = x_n - f(x_n)/f'(x_n)$

Newton's method:

$x_{n+1} = x_n - f(x_n)/f'(x_n)$

General fixed-point theory:

$x_{n+1} = g(x_n)$, where $r = g(r)$ is root of $f(r) = 0$:

$$x_{n+1} - r = g(x_n) - g(r) = [g(r) + g'(\psi)(x_n - r)] - g(r)$$

$$e_{n+1} = g'(\psi)e_n \approx g'(r)e_n$$

Thus if $|g'(r)| < 1$, iteration converges for sufficiently good initial guess. If $g'(r) = 0$, then we have "quadratic" convergence:

$$x_{n+1} - r = g(x_n) - g(r) = [g(r) + g'(r)(x_n - r) + \frac{1}{2}g''(\psi)(x_n - r)^2] - g(r)$$

$$e_{n+1} = \frac{1}{2}g''(\psi)e_n^2 \approx \frac{1}{2}g''(r)e_n^2$$

General theory applied to Newton's method:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

$$g''(x) = \frac{f'(x)^2[f(x)f'''(x) + f'(x)f''(x)] - f(x)f''(x)2f'(x)f''(x)}{f'(x)^4}$$

So, assuming $f'(r) \neq 0$, that is, assuming r is not a multiple root, $g'(r) = 0, g''(r) = \frac{f''(r)}{f'(r)}$ and Newton's method converges quadratically:

$$e_{n+1} \approx \frac{1}{2} g''(r) e_n^2 = \frac{1}{2} \frac{f''(r)}{f'(r)} e_n^2$$

If r is a root of multiplicity m, ie, $f(x) = (x-r)^m q(x)$, where $q(r) \neq 0$ then:

$$g(x) = x - \frac{(x-r)^m q(x)}{(x-r)^m q'(x) + m(x-r)^{m-1} q(x)} = x - \frac{(x-r)q(x)}{(x-r)q'(x) + mq(x)}$$

and

$$g'(x) = 1 - \frac{q(x)}{(x-r)q'(x) + mq(x)} - (x-r)\left[\frac{q(x)}{(x-r)q'(x) + mq(x)}\right]'$$

$$g'(r) = 1 - \frac{q(r)}{mq(r)} = 1 - 1/m$$

So Newton's method still converges, but only linearly.

To calculate "experimental" order of convergence, we assume:

$e_{n+1} \approx M e_n^\alpha$

$e_n \approx M e_{n-1}^\alpha$

$\frac{e_{n+1}}{e_n} \approx \left[\frac{e_n}{e_{n-1}}\right]^\alpha$

$\alpha \approx \frac{log(\frac{e_{n+1}}{e_n})}{log(\frac{e_n}{e_{n-1}})}$