Math 4329, Test I

Name ____________________________

1. a. If $f(x) = e^{-3x}$, find the Taylor polynomial $T_n(x)$ of degree n which matches $f, f', f''...f^{(n)}$ at $a = 0$.

   b. Use the Taylor remainder formula to get a reasonable bound (in terms of n) on the error $|f(x) - T_n(x)|$ at $x = 2$.

2. Computer A stores floating point numbers in a 128-bit word, which includes 1 sign bit, 31 bits for the exponent, and 96 bits for the mantissa. Computer B stores floating point numbers in a 128-bit word, with 1 sign bit, 13 bits for the exponent, and 114 bits for the mantissa.

   a. Which computer can handle larger numbers? Approximately what is the overflow limit for this computer?

   b. Which computer has higher accuracy? Approximately how many significant **decimal** digits of accuracy does this computer have?

3. A root-finder produces approximations $x_3 = 5.01, x_4 = 5.0001, x_5 = 5.000\ 000\ 06$ when applied to $f(x) \equiv x^3 - 25x = 0$. Estimate the experimental order of convergence. What method have we studied that has approximately this order?

4. Consider the iteration $x_{n+1} = g(x_n)$, and suppose that r is a root, $r = g(r)$, and furthermore that $g'(r) = 0$. Show, using a Taylor series expansion, that the error $e_n \equiv x_n - r$ satisfies $e_{n+1} = \frac{1}{2}g''(c_n)e_n^2$, where c_n is between x_n and r. Thus, the method converges quadratically, near the root.

5. a. Newton's method is sometimes used to find $\frac{1}{b}$ by computing the root of $f(x) = b - \frac{1}{x}$. Write the Newton iteration in a form where no divisions are required (thus we can find $\frac{1}{b}$ without doing any divisions).

   b. Use the formula from problem 4 to determine the interval around the root $r = \frac{1}{b}$ in which x_0 must lie, for the Newton iteration of problem 5a to converge. (Hint: $e_{n+1} = [\frac{1}{2}g''(c_n)e_n]e_n$; find values of x_0 for which the quantity in brackets is less than one in absolute value.)