Math 4329, Test II

Name ______________________________

1. a. Suppose $P_3(x)$ is the polynomial of degree 3 which interpolates $f(x) = cos(3x)$ at $x_0 = -1.5h, x_1 = -0.5h, x_2 = 0.5h, x_3 = 1.5h$. Find a reasonable upper bound on the error for $x_0 < x < x_3$. You can be "lazy," that is, bound $|x - x_i|$ for each i by $|x_3 - x_0|$.

   b. [Extra credit] Same question, but now, DON'T be lazy, get the best possible bound on $|q(x)| \equiv |(x-x_0)(x-x_1)(x-x_2)(x-x_3)|$.

2. Find A, B which make the approximation

   $\int_0^h f(x)dx \approx Ahf(0) + Bhf(\frac{2h}{3})$

   as high order as possible. With your choice of A, B, what is the degree of precision, and what is the order of the error (power of h that the global error is proportional to) in this approximation?

3. Consider the linear system:

$$\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1+\epsilon \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

a. Write out the equations for the Jacobi iterative method for solving this system (don't actually do any iterations).

b. Write out the equations for the Gauss-Seidel iterative method for solving this system.

c. True or False: If $\epsilon > 0$, the Jacobi iterative method (3a) will converge for *any* starting vector (x_0, y_0). Give a reason for your answer.

d. Find the condition number of the above matrix (using the L_∞ norm). If you were to solve the above linear system using Gaussian elimination with partial pivoting, would you expect serious roundoff errors, if ϵ is very small?
Hint: The inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4. The following function is a cubic spline for what values of a, b, c?

$$\begin{aligned} s(x) &= 2x^3 - 3x^2 + 3x - 6 \quad for \quad 0 < x \leq 1 \\ &= x^3 + ax^2 + bx + c \quad for \quad 1 < x \leq 2 \end{aligned}$$

5. [Extra Credit] Use Taylor series expansions to determine the error in the approximation

$$u^{iv}(t) \approx \frac{u(t+2h) - 4u(t+h) + 6u(t) - 4u(t-h) + u(t-2h)}{h^4}$$

Hint: expand $u(t + 2h)$, etc, out to the h^6 term.