Math 4329, Final

Name ______________________________

1. a. Let $T_3(x)$ be the Taylor polynomial of degree 3 which matches $f(x), f'(x), f''(x)$ and $f'''(x)$ at $x = 0$, where $f(x) = cos(4x)$. Find the best possible bound on

   $max_{-0.5 \leq x \leq 0.5} |T_3(x) - f(x)| \leq$

   b. Let $L_3(x)$ be the Lagrange polynomial of degree 3 which matches $f(x)$ at $x = -2, -0.5, 0.5$ and 2, where $f(x) = cos(4x)$. Find the best possible bound on

   $max_{-0.5 \leq x \leq 0.5} |L_3(x) - f(x)| \leq$

2. a. A root finder gives consecutive errors of $e_8 = 10^{-3}, e_9 = 10^{-5}, e_{10} = 10^{-11}$. Estimate the order of the method.

   b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-11} when $h = 10^{-4}$. Estimate the order of the method.

   c. A differential equation solver gives an answer $u(1) = 1.08888$ when $h = 0.1$, and $u(1) = 1.00666$ when $h = 0.01$, and $u(1) = 1.00600$ when $h = 0.001$. Estimate the order of the method.

3. Use the inverse power method to determine the smallest eigenvalue of

$$A = \begin{bmatrix} \frac{4}{6} & \frac{-2}{6} \\ \\ \frac{-5}{6} & \frac{1}{6} \end{bmatrix}$$

Start the iteration with $(x_0, y_0) = (3, 8)$.

4. $x^2 + xy^3 = 9 - 3y$
   $3x^2y - y^3 = 4 + 2x$

   Do one iteration of Newton's method, to find a root of this system, starting from $(x_0, y_0) = (0, 0)$.

5. Take one step of a third order Taylor series method (Euler is the first order Taylor method) to approximate the solution of the following problem, at $t = 1.01$:

   $u' = 2tu$
   $u(1) = 1$

6. Will the iteration $x_{n+1} = 2 - \frac{3}{2}x_n + \frac{1}{2}x_n^3$ converge when x_0 is sufficiently close to the root $r = 1$? If so, what is the order of convergence? (Justify your answer theoretically, without actually iterating the formula.)

7. a. Write the third order differential equation $u''' - 3u'' - u = t^2$ as a system of three first order equations, that is, in the form:

   $u' = f(t, u, v, w) =$

   $v' = g(t, u, v, w) =$

   $w' = h(t, u, v, w) =$

   b. Now write out the formulas for $u_{n+1}, v_{n+1}, w_{n+1}$ for Euler's method applied to this system of first order equations:

   $u_{n+1} =$

   $v_{n+1} =$

   $w_{n+1} =$

8. Write $\frac{\sqrt{4+x}-2}{x}$ in a form where there is no serious problem with roundoff, when $x \approx 0$.