Math 4329, Final

Name ______________________________

Do the true/false problem (last problem) and 8 of the other 9 problems. Clearly mark which problem NOT to grade.

1. Let $T_4(x)$ be the Taylor polynomial of degree 4 which matches $f(x), f'(x), f''(x), f'''(x)$ and $f^{iv}(x)$ at $x = 2$, where $f(x) = e^{x/3}$. Find a reasonable bound on

   $max_{0 \le x \le 4} |T_4(x) - f(x)| \le$

2. Let $p_4(x)$ be the fourth degree polynomial which satisfies $p_4(x_i) = f(x_i),\ i = 0, 1, 2, 3, 4$, where $f(x) = e^{x/3}$. Give a formula for the error $f(x) - p_4(x)$ at an arbitrary point x.

3. Determine the degree of precision and (global) order of the quadrature rule:

   $\int_0^h f(x)dx \approx \frac{h}{8} f(0) + \frac{3h}{8} f(\frac{h}{3}) + \frac{3h}{8} f(\frac{2h}{3}) + \frac{h}{8} f(h)$

4. Use the power method to approximate the largest eigenvalue and the associated eigenvector of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 8 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

5. $f_1(x, y) = x + y + 3$
   $f_2(x, y) = 2x + y$

   Do two iterations of Newton's method, to find a root of $f_1 = f_2 = 0$, starting from $(x_0, y_0) = (1, 1)$.

6. Consider the problem

   $u' = -u^2$
   $u(1) = 2$

   Take one step of a second order Taylor series method with $h = 0.01$ to approximate $u(1.01)$.

7. a. Write the third order differential equation $u''' - 3u'' - u' = t^2$ as a system of three first order equations, that is, in the form:

$$
\begin{aligned}
u' &= f(t,u,v,w) = \\
v' &= g(t,u,v,w) = \\
w' &= h(t,u,v,w) =
\end{aligned}
$$

b. Now write out the formulas for $u_{n+1}, v_{n+1}, w_{n+1}$ for Euler's method applied to this system of first order equations:

$$
\begin{aligned}
u_{n+1} &= \\
v_{n+1} &= \\
w_{n+1} &=
\end{aligned}
$$

8. Will the iteration $x_{n+1} = 4x_n(1 - x_n)$ converge when x_0 is sufficiently close to the root $r = \frac{3}{4}$? (Justify your answer theoretically, without actually iterating the formula.)

9. Will the following iteration converge (to something)? (Justify your answer theoretically, without actually iterating the equations.)

$$
\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.75 \\ -0.75 & 0.5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}
$$

10. True or False:

a. Serious roundoff error can usually be traced to operations involving multiplication or division.

b. The experimental order of convergence is $O(h^3)$ if a quadrature rule yields errors of 0.0032 when $h = 0.01$ and 0.0002 when $h = 0.0025$.

c. If a root-finder gives three consecutive errors of $10^{-5}, 10^{-7}$ and 10^{-11}, the experimental order is quadratic (2)?

d. Of all quadrature rules with n sample points per strip, the Gauss n-point formula has the highest order of accuracy.

e. A disadvantage of the Runge-Kutta methods is that they require several starting values.

f. It is easier to vary the stepsize for a Runge-Kutta method than an Adams multistep method.

g. Taylor series methods are not widely used by general purpose ODE solvers because they require that the user supply derivatives of $f(t, u)$.

h. If $f(r) = f'(r) = 0$, Newton's method will converge **quadratically** to r if x_0 is sufficiently close to the root r.

i. Euler's method is equivalent to a first order Taylor series method.

j. The Gauss-Seidel iterative method (for $Ax = b$) is generally faster than the Jacobi method.

k. The Jacobi iterative method (for $Ax = b$) converges only if the matrix is diagonal-dominant.

l. Gaussian elimination, when applied to a general N by N linear system, requires $O(N^4)$ arithmetic operations.

m. If $s(x)$ is a cubic spline, then s, s' and s'' must be continuous everywhere.

n. If a quadrature method is exact for all polynomials of degree n, its global error is $O(h^{n+1})$ for general smooth functions.

o. If Gaussian elimination is used with NO pivoting, large roundoff errors may result even if A is well-conditioned.

p. If Gaussian elimination is used with partial pivoting, the solution is usually very accurate even if A is ill-conditioned.