Math 4329, Final

Name ______________________________

1. a. Let $T_3(x)$ be the Taylor polynomial of degree 3 which matches $f(x), f'(x), f''(x)$ and $f'''(x)$ at $x = 0$, where $f(x) = cos(4x)$. Find the best possible bound on

   $max_{-2\leq x\leq 2}|T_3(x) - f(x)| \leq$

   b. Let $L_3(x)$ be the Lagrange polynomial of degree 3 which matches $f(x)$ at $x = -3, -2, 2$ and 3, where $f(x) = cos(4x)$. Find the best possible bound on

   $max_{-2\leq x\leq 2}|L_3(x) - f(x)| \leq$

2. a. A root finder gives consecutive errors of $e_8 = 10^{-5}, e_9 = 10^{-6}, e_{10} = 10^{-11}$. Estimate the order of the method.

   b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-13} when $h = 10^{-4}$. Estimate the order of the method.

3. Use the inverse power method to determine the smallest eigenvalue of

$$A = \begin{bmatrix} \frac{-4}{6} & \frac{2}{6} \\ \\ \frac{5}{6} & \frac{-1}{6} \end{bmatrix}$$

Start the iteration with $(x_0, y_0) = (3, 8)$, and do 2 iterations.

4. $x^2 + xy^3 + 3y = 9$
$3x^2y - y^3 - 2x = 4$

Do one iteration of Newton's method, to find a root of this system, starting from $(x_0, y_0) = (0, 0)$.

5. Take one step of a second order Taylor series method (Euler is the first order Taylor method) with $h = 0.001$ to approximate the solution of the following problem, at $t = 0.001$:

$u' = 4t + u^3$
$u(0) = 2$

6. Will the iteration $x_{n+1} = 2 - \frac{3}{2}x_n + \frac{1}{2}x_n^3$ converge when x_0 is sufficiently close to the root $r = 1$? If so, what is the order of convergence? (Justify your answer theoretically, without actually iterating the formula.)

7. a. Reduce

$$y'' = 3y'y - e^t z'$$
$$z'' = z'z - \sqrt{y}$$

to a system of 4 first order equations. The right hand sides must involve only $t, u1, u2, u3, u4$.

$u1' =$

$u2' =$

$u3' =$

$u4' =$

b. Now write out the formulas for $u1_{n+1}, u2_{n+1}, u3_{n+1}, u4_{n+1}$ for Euler's method applied to this system of first order equations:

$u1_{n+1} =$

$u2_{n+1} =$

$u3_{n+1} =$

$u4_{n+1} =$

8. Write $\frac{(8+x)^{\frac{1}{3}}-2}{x}$ in a form where there is no serious problem with round-off, when $x \approx 0$. (Hint: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.)

9. Consider the linear system:

$$\begin{bmatrix} 1 & 2.0000001 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

a. Write out the equations for the Jacobi iterative method for solving this system (don't actually do any iterations).

b. Write out the equations for the Gauss-Seidel iterative method for solving this system.

c. Calculate the condition number for this matrix. Hint, if:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} /(ad - bc)$$

If machine precision is $\epsilon = 10^{-16}$, about how many significant figures would you expect in the solution, if Gauss elimination with partial pivoting is used to solve this linear system?