Math 4329, Final

Name ____________________________

1. a. Let $T_4(x)$ be the Taylor polynomial of degree 4 which matches $f(x), f'(x), f''(x), f'''(x)$ and $f^{iv}(x)$ at $x = 1$, where $f(x) = e^{-2x}$. Find a reasonable bound on

   $|T_4(1.2) - f(1.2)| \leq$

   b. Let $L_4(x)$ be the Lagrange polynomial of degree 4 which matches $f(x)$ at $x = 0, 0.5, 1, 1.5$ and 2, where $f(x) = e^{-2x}$. Find a reasonable bound on

   $|L_4(1.2) - f(1.2)| \leq$

2. If $a = 0, b = 1000$ and $f(a)$ and $f(b)$ have opposite signs, how many bisection iterations are required to find a root between a and b to an accuracy of 10^{-8}?

3. Reduce

$$y'' = 3y'y - e^t z$$
$$z'' = z'z - \sqrt{y}$$

to a system of 4 first order equations. The right hand sides must involve only t, u_1, u_2, u_3, u_4. The left hand sides must be u_1', u_2', u_3', u_4' respectively.

4. a. A root finder gives consecutive errors of $e_8 = 10^{-3}, e_9 = 10^{-5}, e_{10} = 10^{-12}$. Estimate the order of the method.

   b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-12} when $h = 10^{-4}$. Estimate the order of the method.

   c. A differential equation solver gives an answer $u(1) = 1.08888$ when $h = 0.1$, and $u(1) = 1.00666$ when $h = 0.01$, and $u(1) = 1.00600$ when $h = 0.001$. Estimate the order of the method.

5. Find A, B which make the approximation

   $\int_0^h f(x)dx \approx Ahf(0.2h) + Bhf(0.9h)$

   as high order as possible.

6. Use the inverse power method to determine the smallest eigenvalue of

   $$A = \begin{bmatrix} \frac{4}{6} & \frac{-2}{6} \\ \frac{-5}{6} & \frac{1}{6} \end{bmatrix}$$

   Start the iteration with $(x_0, y_0) = (3, 8)$.

7. Will the iteration $x_{n+1} = 2 - \frac{3}{2}x_n + \frac{1}{2}x_n^3$ converge when x_0 is sufficiently close to the root $r = 1$? If so, what is the order of convergence? (Justify your answer theoretically, without actually iterating the formula.)

8. If the second order Taylor series method (one more term than Euler's method) is used to solve $u' = t^2\sqrt{1+u^2}$, write u_{n+1} in terms of h, t_n and u_n only. $(t_n = nh, u_n \approx u(t_n))$

9. Do **two** iterations of Newton's method, starting from $(0, 0)$, to solve:

   $f(x, y) = 2x + y - 4 = 0$
   $g(x, y) = -6x + y + 4 = 0$

10. Will the following iteration converge (to something)? **Justify** your answer without actually iterating.

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.75 \\ -0.75 & 0.5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$