Math 4329, Final (e)

Name ____Key____

1. If the second order Taylor series method (one more term than Euler's method) is used to solve $u' = t^2 + u^2$, write u_{n+1} in terms of h, t_n and u_n only. ($t_n = nh, u_n \approx u(t_n)$)

4

$$u' = t^2 + u^2 \qquad u'' = 2t + 2u(u^2 + t^2)$$

$$u_{n+1} = u_n + h(t_n^2 + u_n^2) + \frac{h^2}{2}\left[2t_n + 2u_n(t_n^2 + u_n^2)\right]$$

2. a. Let $T_2(x)$ be the Taylor polynomial of degree 2 which matches $f(x), f'(x)$ and $f''(x)$ at $x = 0$, where $f(x) = \frac{1}{1+2x}$. Find a reasonable bound on

$$f = (1+2x)^{-1}$$
$$f' = -2(1+2x)^{-2}$$
$$f'' = 8(1+2x)^{-3}$$
$$f''' = -48(1+2x)^{-4}$$

$$max_{-0.1 \le x \le 0.1} |T_2(x) - f(x)| \le \left|\frac{f'''(\xi)}{6} x^3\right|$$

$$\le \left|\frac{48}{(1+2\xi)^4} \frac{1}{6} x^3\right| \le \frac{48}{(0.8)^4} \frac{(0.1)^3}{6} = 0.0195$$

b. Let $L_2(x)$ be the Lagrange polynomial of degree 2 which matches $f(x)$ at $x = -0.1, 0.0$ and 0.1, where $f(x) = \frac{1}{1+2x}$. Find a reasonable bound on

6

$$max_{-0.1 \le x \le 0.1} |L_2(x) - f(x)| \le \left|\frac{f'''(\xi)}{6}(x+0.1)x(x-0.1)\right|$$

$$\le \left|\frac{48}{(1+2\xi)^4} \frac{1}{6}(x+0.1)x(x-0.1)\right| \le \frac{48}{(0.8)^4} \frac{1}{6}(0.2)^3$$

$$= 0.156$$

3. a. A root finder gives consecutive errors of $e_5 = 10^{-2}, e_6 = 10^{-4}, e_7 = 10^{-11}$. Estimate the order of the method.

$$\frac{10^{-11} = M(10^{-4})^\alpha}{10^{-4} = M(10^{-2})^\alpha} \qquad 10^7 = 10^{2\alpha} \qquad \alpha = 3.5$$

b. A quadrature method gives an error of 10^{-7} when $h = 0.001$ and 10^{-11} when $h = 0.0001$. Estimate the order of the method.

$$\frac{10^{-7} = M(.001)^\alpha}{10^{-11} = M(.0001)^\alpha} \qquad 10^4 = 10^\alpha \qquad \alpha = 4$$

6

c. A differential equation solver gives an answer $u(1) = 0.98888$ when $h = 0.1$, and $u(1) = 0.90666$ when $h = 0.01$, and $u(1) = 0.90600$ when $h = 0.001$. Estimate the order of the method.

$$0.98888 - I = M(.1)^\alpha$$
$$0.90666 - I = M(.01)^\alpha$$
$$0.90600 - I = M(.001)^\alpha$$

$$\frac{.08222 = M[(.1)^\alpha - (.01)^\alpha]}{.00066 = M[(.01)^\alpha - (.001)^\alpha]}$$

$$125 = 10^\alpha \qquad \alpha = 2.1$$

4. Use the inverse power method to find the smallest (in absolute value) eigenvalue of A, if

$$A^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \lambda = \begin{Bmatrix} 1.89 \\ 2.00 \\ 20.11 \end{Bmatrix}$$

Start with $(1, 10, 1)$ and do 3 iterations.

4

$$\begin{pmatrix} 1 \\ 10 \\ 1 \end{pmatrix} \to \begin{pmatrix} 12 \\ 202 \\ 12 \end{pmatrix} \to \begin{pmatrix} 226 \\ 4064 \\ 226 \end{pmatrix} \to \begin{pmatrix} 4516 \\ 81732 \\ 4516 \end{pmatrix} \qquad \text{ratios} \quad \begin{matrix} 19.98 \\ 20.11 \\ 19.98 \end{matrix}$$

$$\lambda_{max}(A^{-1}) = 20 \qquad \lambda_{min}(A) = 0.05$$

5. Reduce

$$y'' = y'y - tz$$
$$z''' = z''z - y$$

$u_1 = y$
$u_2 = y'$
$u_3 = z$
$u_4 = z'$
$u_5 = z''$

to a system of 5 first order equations. The right hand sides must involve only $t, u_1, u_2, u_3, u_4, u_5$. The left hand sides must be $u_1', u_2', u_3', u_4', u_5'$ respectively.

4

$$\begin{cases} u_1' = u_2 \\ u_2' = u_2 u_1 - t u_3 \\ u_3' = u_4 \\ u_4' = u_5 \\ u_5' = u_5 u_3 - u_1 \end{cases}$$

6. Do one iteration of Newton's method, starting from $(0,0)$, to solve:

$f(x,y) = 2x^2 + y - 1 = 0$
$g(x,y) = -2x + y^2 + 1 = 0$

4

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} 4x & 1 \\ -2 & 2y \end{pmatrix}_n^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

7. Write $(1+x)^{1/3} - 1$ in a form where there is no serious problem with roundoff, when $x \approx 0$. (Hint: $a^3 - b^3 = (a^2 + ab + b^2)(a - b)$)

$$(1+x)^{\frac{1}{3}} - 1 = \frac{(1+x) - 1^3}{(1+x)^{\frac{2}{3}} + (1+x)^{\frac{1}{3}} + 1} = \frac{x}{(1+x)^{\frac{2}{3}} + (1+x)^{\frac{1}{3}} + 1}$$

4

8. How should A, B, r be chosen to make the approximation:

$\int_{-1}^{1} f(x)dx \approx Af(-r) + Bf(0) + Af(r)$

as high degree of precision as possible?

4

$f=1 \quad 2 = \int_{-1}^{1} 1\,dx = A + B + A \qquad 2A+B=2$

$f=x^2 \quad \frac{2}{3} = \int_{-1}^{1} x^2 dx = Ar^2 + Ar^2 \qquad 2Ar^2 = \frac{2}{3}$

$f=x^4 \quad \frac{2}{5} = \int_{-1}^{1} x^4 dx = Ar^4 + Ar^4 \qquad 2Ar^4 = \frac{2}{5}$

$r^2 = \frac{2/5}{2/3} = \frac{3}{5} \qquad r = \sqrt{\frac{3}{5}} \qquad B = \frac{8}{9}$

$A = \frac{1}{3r^2} = \frac{5}{9} = A$

9. Will the iteration $x_{n+1} = \frac{1}{2}(x_n + \frac{5}{x_n})$ converge to the root $\sqrt{5}$, if the starting guess is sufficiently good? Justify your answer theoretically.

4

$g(x) = \frac{1}{2}(x + 5x^{-1})$

$g'(x) = \frac{1}{2} - \frac{5}{2}x^{-2}$

$g'(\sqrt{5}) = \frac{1}{2} - \frac{5}{2}\frac{1}{5} = 0$ yes converges

Math 4329, Final (f)

Name Key

1. a. Let $T_4(x)$ be the Taylor polynomial of degree 4 which matches $f(x), f'(x), f''(x), f'''(x)$ and $f^{iv}(x)$ at $x = 0$, where $f(x) = cos(x/6)$. Find the best possible bound on

$-\pi \le \xi \le \pi$

4 $max_{-\pi\le x\le\pi}|T_4(x) - f(x)| \le \left|\frac{f^{v}(\xi)}{5!}x^5\right| \le \frac{(\frac{1}{6})^5 \sin(\frac{\xi}{6})}{120}\pi^5$

$\le (\frac{1}{6})^5 \frac{\frac{1}{2}}{120}\pi^5 = 1.6\cdot10^{-4}$

b. Let $L_4(x)$ be the Lagrange polynomial of degree 4 which matches $f(x)$ at $x = -\pi, -0.5, 0, 0.5$ and π, where $f(x) = cos(x/6)$. Find the best possible bound on

4 $|L_4(1) - f(1)| \le \left|\frac{f^{v}(\xi)}{5!}(1+\pi)(1+0.5)\,1\,(1-0.5)(1-\pi)\right|$

$\le \frac{(\frac{1}{6})^5 \sin\frac{\xi}{6}}{120}\,6.65 \le \frac{(\frac{1}{6})^5\frac{1}{2}}{120}\,6.65 = 3.56\cdot10^{-6}$

2. a. A root finder gives consecutive errors of $e_5 = 10^{-3}, e_6 = 10^{-5}, e_7 = 10^{-10}$. Estimate the order of the method. $10^{-10} = M(10^{-5})^\alpha$

2 $10^{-5} = (10^{-2})^\alpha$ $\quad \alpha = 2.5 \quad$ $10^{-5} = M(10^{-3})^\alpha$

b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-9} when $h = 10^{-3}$. Estimate the order of the method.

$10^{-5} = M(10^{-2})^\alpha$

2 $10^{-9} = M(10^{-3})^\alpha$ $\quad 10^4 = 10^\alpha \quad \alpha = 4$

c. A differential equation solver gives an answer $u(1) = 2.18888$ when $h = 0.1$, and $u(1) = 2.10666$ when $h = 0.01$, and $u(1) = 2.10600$ when $h = 0.001$. Estimate the order of the method.

$2.18888 - I = M(.1)^\alpha$ $\quad 0.08222 = M\left((.1)^\alpha - (.01)^\alpha\right)$

3 $2.10666 - I = M(.01)^\alpha$

$2.10600 - I = M(.001)^\alpha$ $\quad 0.00066 = M\left((.01)^\alpha - (.001)^\alpha\right)$

$125 = 10^\alpha \quad \alpha = 2.1$

3. Use the power method to find the largest (in absolute value) eigenvalue of

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

What is the corresponding eigenvector?

5

$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $x_1 = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}$ $x_2 = \begin{pmatrix} 11 \\ 21 \\ 11 \end{pmatrix}$ $x_3 = \begin{pmatrix} 43 \\ 85 \\ 43 \end{pmatrix}$

4 4

$\lambda_{max} = 4$ $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

4. $x^2 + xy^3 = 9 - 3y$
$3x^2y - y^3 = 4 + 2x$

Do one iteration of Newton's method, to find a root of this system, starting from $(x_0, y_0) = (0, 0)$.

5

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} 2x + y^3 & 3xy^2 + 3 \\ 6xy - 2 & 3x^2 - 3y^2 \end{pmatrix}^{-1}_{(0,0)} \begin{pmatrix} -9 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -9 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

5. Will the iteration $x_{n+1} = x_n + sin(x_n)$ converge when x_0 is sufficiently close to the root $r = \pi$? If so, what is the order of convergence? (Justify your answer theoretically, without actually iterating the formula.)

5

$g(x) = x + \sin x$ $\quad g'(\pi) = 1 - 1 = 0$

$g'(x) = 1 + \cos x$ $\quad g''(\pi) = 0$ $\quad$ Converges, order $= 3$

$g''(x) = -\sin x$ $\quad g'''(\pi) = 1$

$g'''(x) = -\cos x$

6. a. Write the third order differential equation $u''' - 5u' - u = t^4$ as a system of three first order equations, that is, in the form:

$u' = f(t, u, v, w) =$ v

$v' = g(t, u, v, w) =$ w

$w' = h(t, u, v, w) =$ $5v + u + t^4$

b. Now write out the formulas for $u_{n+1}, v_{n+1}, w_{n+1}$ for Euler's method applied to this system of first order equations:

5

$u_{n+1} =$ $u_n + h v_n$

$v_{n+1} =$ $v_n + h w_n$

$w_{n+1} =$ $w_n + h(5v_n + u_n + t_n^4)$

7. If the third order Taylor series method (two more terms than Euler's method) is used to solve $u' = t(1 + u)$, write u_{n+1} in terms of h, t_n and u_n only. ($t_n = nh, u_n \approx u(t_n)$)

5

$u' = t(1+u)$

$u'' = (1+u) + tu' = (1+t^2)(1+u)$

$u''' = (1+t^2)u' + 2t(1+u) = (3t + t^3)(1+u)$

$u_{n+1} = u_n + h\, t_n(1+u_n) + \frac{h^2}{2}(1+t_n^2)(1+u_n) + \frac{h^3}{6}(3t_n + t_n^3)(1+u_n)$

Math 4329, Final (9)

Name Key

1. a. Let $T_3(x)$ be the Taylor polynomial of degree 3 which matches $f(x), f'(x), f''(x)$ and $f'''(x)$ at $x = 0$, where $f(x) = cos(x/3)$. Find the best possible bound on

2

$$max_{-0.5\le x\le 0.5}|T_3(x) - f(x)| \le \left|\frac{f^{iv}(\xi)}{4!}x^4\right| \le \frac{\left(\frac{1}{3}\right)^4}{24}\left(\frac{1}{2}\right)^4 = 3.2\cdot 10^{-5}$$

b. Let $L_3(x)$ be the Lagrange polynomial of degree 3 which matches $f(x)$ at $x = -2, -1, 1$ and 2, where $f(x) = cos(x/3)$. Find the best possible bound on

2

$$|L_3(0) - f(0)| \le \left|\frac{f^{iv}(\xi)}{24}2(1)(1)(2)\right| \le \frac{1}{6}\left(\frac{1}{3}\right)^4 = 2.0\cdot 10^{-3}$$

2. a. A root finder gives consecutive errors of $e_8 = 10^{-3}, e_9 = 10^{-5}, e_{10} = 10^{-11}$. Estimate the order of the method.

2

$$10^{-5} = M(10^{-3})^\alpha \qquad 10^{6} = (10^2)^\alpha \qquad \alpha = 3$$
$$10^{-11} = M(10^{-5})^\alpha$$

b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-11} when $h = 10^{-4}$. Estimate the order of the method.

2

$$10^{-5} = M(10^{-2})^\alpha \qquad 10^{6} = (10^2)^\alpha \qquad \alpha = 3$$
$$10^{-11} = M(10^{-4})^\alpha$$

3. Use the inverse power method to determine the smallest eigenvalue of

$$A = \begin{bmatrix} \frac{4}{6} & \frac{-2}{6} \\ \frac{-5}{6} & \frac{1}{6} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix}$$

Start the iteration with $(x_0, y_0) = (3, 8)$.

2

$$\begin{pmatrix} 3 \\ 8 \end{pmatrix} \to \begin{pmatrix} -19 \\ -47 \end{pmatrix} \to \begin{pmatrix} 113 \\ 283 \end{pmatrix} \qquad \text{ratio} \simeq -6 \qquad \lambda_{min} = -\frac{1}{6}$$

4. Take one step of a third order Taylor series method (Euler is the first order Taylor method) to approximate the solution of the following problem, at $t = 1.01$:

$u' = 2tu$
$u(1) = 1$

$$u' = 2tu = 2$$
$$u'' = 2tu' + 2u = 2(2) + 2(1) = 6$$
$$u''' = 2tu'' + 4u' = 2(1)(6) + 4(2) = 20$$

2

$$u(1+h) \doteq u(1) + u'(1)h + u''(1)\frac{h^2}{2} + u'''(1)\frac{h^3}{6}$$
$$= 1 + 2h + 6\frac{h^2}{2} + \frac{20}{6}h^3 = 1.02030337$$

5. Do one iteration of Newton's method, starting from $(0, 0)$, to solve:

$f(x, y) = 2x^2 + y - 1 = 0$
$g(x, y) = -2x + y^2 + 1 = 0$

2

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} 4x_n & 1 \\ -2 & 2y_n \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

6. Write $\sqrt{1+x} - 1$ in a form where there is no serious problem with roundoff, when $x \approx 0$.

$$\frac{(\sqrt{1+x} - 1)(\sqrt{1+x}+1)}{\sqrt{1+x}+1} = \frac{1+x-1}{1+\sqrt{1+x}} = \frac{x}{1+\sqrt{1+x}}$$

2

7. Determine the degree of precision of the quadrature rule:
$\int_0^h f(x)dx \approx \frac{h}{4}f(0) + \frac{3h}{4}f(\frac{2h}{3})$

$f=1$ $\quad h = \int_0^h 1\,dx = \frac{h}{4} + \frac{3h}{4} = h$ ✓

$f=x$ $\quad \frac{h^2}{2} = \int_0^h x\,dx = \frac{3h}{4}\frac{2h}{3} = \frac{1}{2}h^2$ ✓

2 $\quad f=x^2$ $\quad \frac{h^3}{3} = \int_0^h x^2 dx = \frac{3h}{4}\left(\frac{2h}{3}\right)^2 = \frac{h^3}{3}$ ✓ $\quad$ d.o.p. = 2

$f=x^3$ $\quad \frac{h^4}{4} = \int_0^h x^3 dx = \frac{3h}{4}\left(\frac{2h}{3}\right)^3 = \frac{2h^4}{9}$ no

8. Will the iteration $x_{n+1} = 2 - \frac{3}{2}x_n + \frac{1}{2}x_n^3$ converge to the root 1, if the starting guess is sufficiently good? **Justify** your answer.

2

$g(x) = 2 - \frac{3}{2}x + \frac{1}{2}x^3$ $\quad g(1) = 1$

$g'(x) = -\frac{3}{2} + \frac{3}{2}x^2$ $\quad g'(1) = 0$ so yes

Math 4329, Final (h)

Name Key

Do the true/false problem (last problem) and 8 of the other 9 problems. Clearly mark which problem NOT to grade.

1. Let $T_4(x)$ be the Taylor polynomial of degree 4 which matches $f(x), f'(x), f''(x), f'''(x)$ and $f^{iv}(x)$ at $x = 2$, where $f(x) = e^{x/3}$. Find a reasonable bound on

$max_{0\leq x\leq 4}|T_4(x) - f(x)| \leq \left|\frac{f^{v}(\xi)}{5!}(x-2)^5\right| \leq \frac{\frac{1}{3^5}e^{(\xi/3)}2^5}{120}$

4

$\leq \left(\frac{2}{3}\right)^5 \frac{e^{\frac{4}{3}}}{120} = 0.00416$

2. Let $p_4(x)$ be the fourth degree polynomial which satisfies $p_4(x_i) = f(x_i),\ i = 0, 1, 2, 3, 4$, where $f(x) = e^{x/3}$. Give a formula for the error $f(x) - p_4(x)$ at an arbitrary point x.

4

$f(x) - p_4(x) = \frac{f^{v}(\xi)}{5!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$

$= \frac{1}{243}\frac{e^{(\xi/3)}}{120}(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$

3. Determine the degree of precision and (global) order of the quadrature rule:

$\int_0^h f(x)dx \approx \frac{h}{8}f(0) + \frac{3h}{8}f(\frac{h}{3}) + \frac{3h}{8}f(\frac{2h}{3}) + \frac{h}{8}f(h)$

$f=1 \quad h \stackrel{?}{=} \frac{h}{8} + \frac{3h}{8} + \frac{3h}{8} + \frac{h}{8} = h$ ✓

$x \quad \frac{1}{2}h^2 \stackrel{?}{=} \frac{3h}{8}\frac{h}{3} + \frac{3h}{8}\frac{2h}{3} + \frac{h}{8}h = \frac{h^2}{2}$ ✓

4

$x^2 \quad \frac{1}{3}h^3 \stackrel{?}{=} \frac{3h}{8}\frac{h^2}{9} + \frac{3h}{8}\frac{4h^2}{9} + \frac{h}{8}h^2 = \frac{1}{3}h^3$ ✓

$x^3 \quad \frac{1}{4}h^4 \stackrel{?}{=} \frac{3h}{8}\frac{h^3}{27} + \frac{3h}{8}\frac{8h^3}{27} + \frac{h}{8}h^3 = \frac{1}{4}h^4$ ✓

$x^4 \quad \frac{1}{5}h^5 \neq \frac{3h}{8}\frac{h^4}{81} + \frac{3h}{8}\frac{16h^4}{81} + \frac{h}{8}h^4 = \frac{33}{162}h^5$

$d.o.p = 3 \quad O(h^4)$

4. Use the power method to approximate the largest eigenvalue and the associated eigenvector of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 8 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad x_1 = \begin{pmatrix} 2 \\ 10 \\ 2 \end{pmatrix} \quad x_2 = \begin{pmatrix} 12 \\ 84 \\ 12 \end{pmatrix}$$

4

$$x_3 = \begin{pmatrix} 96 \\ 696 \\ 96 \end{pmatrix} \quad x_4 = \begin{pmatrix} 792 \\ 5760 \\ 792 \end{pmatrix}$$

8.25 8.27 8.28

$$\lambda \approx 8.25 \qquad z = \begin{pmatrix} 0.1375 \\ 1 \\ 0.1375 \end{pmatrix}$$

5. $f_1(x, y) = x + y + 3$
$f_2(x, y) = 2x + y$

Do two iterations of Newton's method, to find a root of $f_1 = f_2 = 0$, starting from $(x_0, y_0) = (1, 1)$.

4

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \underbrace{\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}}_{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -6 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

6. Consider the problem

$$u'' = -2uu' = -2u(-u^2) = 2u^3$$

$u' = -u^2$
$u(1) = 2$

4

Take one step of a second order Taylor series method with $h = 0.01$ to approximate $u(1.01)$.

$$u(1+h) \cong u(1) + u'(1)h + \tfrac{1}{2}u''(1)\tfrac{h^2}{2}$$

$u(1) = 2$
$u'(1) = -4$
$u''(1) = 16$

$$= 2 - 4h + 16\frac{h^2}{2} = 2 - 0.04 + .0008$$

$$1.9608$$

7. a. Write the third order differential equation $u''' - 3u'' - u' = t^2$ as a system of three first order equations, that is, in the form:

$$
\begin{aligned}
u' &= f(t,u,v,w) = V \\
v' &= g(t,u,v,w) = W \\
w' &= h(t,u,v,w) = 3W + V + t^2
\end{aligned}
$$

4

b. Now write out the formulas for $u_{n+1}, v_{n+1}, w_{n+1}$ for Euler's method applied to this system of first order equations:

$$
\begin{aligned}
u_{n+1} &= u_n + h V_n \\
v_{n+1} &= V_n + h w_n \\
w_{n+1} &= w_n + h(3w_n + V_n + t_n^2)
\end{aligned}
$$

8. Will the iteration $x_{n+1} = 4x_n(1 - x_n)$ converge when x_0 is sufficiently close to the root $r = \frac{3}{4}$? (Justify your answer theoretically, without actually iterating the formula.)

4

$g(x) = 4x(1-x) = 4x - 4x^2$

$g'(x) = 4 - 8x$

$g'(\frac{3}{4}) = 4 - 8(\frac{3}{4}) = -2$

no

9. Will the following iteration converge (to something)? (Justify your answer theoretically, without actually iterating the equations.)

$$
\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.75 \\ -0.75 & 0.5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}
$$

4

$\det \begin{pmatrix} 0.5-\lambda & 0.75 \\ -0.75 & 0.5-\lambda \end{pmatrix} = 0$

$\lambda = 0.5 \pm 0.75 i$

$|\lambda| = \sqrt{.8125} = .90$

yes

10. True or False:

F a. Serious roundoff error can usually be traced to operations involving multiplication or division.

F b. The experimental order of convergence is $O(h^3)$ if a quadrature rule yields errors of 0.0032 when $h = 0.01$ and 0.0002 when $h = 0.0025$.

True c. If a root-finder gives three consecutive errors of $10^{-5}, 10^{-7}$ and 10^{-11}, the experimental order is quadratic (2)?

True d. Of all quadrature rules with n sample points per strip, the Gauss n-point formula has the highest order of accuracy.

F e. A disadvantage of the Runge-Kutta methods is that they require several starting values.

8

True f. It is easier to vary the stepsize for a Runge-Kutta method than an Adams multistep method.

True g. Taylor series methods are not widely used by general purpose ODE solvers because they require that the user supply derivatives of $f(t, u)$.

F h. If $f(r) = f'(r) = 0$, Newton's method will converge **quadratically** to r if x_0 is sufficiently close to the root r.

True i. Euler's method is equivalent to a first order Taylor series method.

True j. The Gauss-Seidel iterative method (for $Ax = b$) is generally faster than the Jacobi method.

F k. The Jacobi iterative method (for $Ax = b$) converges only if the matrix is diagonal-dominant.

F l. Gaussian elimination, when applied to a general N by N linear system, requires $O(N^4)$ arithmetic operations.

True m. If $s(x)$ is a cubic spline, then s, s' and s'' must be continuous everywhere.

True n. If a quadrature method is exact for all polynomials of degree n, its global error is $O(h^{n+1})$ for general smooth functions.

True o. If Gaussian elimination is used with NO pivoting, large roundoff errors may result even if A is well-conditioned.

F p. If Gaussian elimination is used with partial pivoting, the solution is usually very accurate even if A is ill-conditioned.

Math 4329, Final (λ)

Name Key

1. Use the power method to find the largest (in absolute value) eigenvalue of

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 10 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Start with $(1,5,1)$ and do 3 iterations. What is the corresponding eigenvector?

4

$$\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \to \begin{pmatrix} 6 \\ 52 \\ 6 \end{pmatrix} \to \begin{pmatrix} 58 \\ 532 \\ 58 \end{pmatrix} \xrightarrow{10.2} \begin{pmatrix} 590 \\ 5436 \\ 590 \end{pmatrix}$$

10.2 $\quad$ 10.2

$\lambda \simeq 10.2$

$z = \begin{pmatrix} 0.108 \\ 1 \\ 0.108 \end{pmatrix}$

2. a. Write the third order differential equation $u''' + 4u'' + 5u' + 2u = 2t^2 + 10t + 8$ as a system of three first order equations, that is, in the form:

3

$u' = f(t,u,v,w) = V$

$v' = g(t,u,v,w) = W$

$w' = h(t,u,v,w) = -4w - 5v - 2u + 2t^2 + 10t + 8$

b. Now write out the formulas for $u_{n+1}, v_{n+1}, w_{n+1}$ for Euler's method applied to this system of first order equations:

3

$u_{n+1} = u_n + h v_n$

$v_{n+1} = v_n + h w_n$

$w_{n+1} = w_n + h(-4w_n - 5v_n - 2u_n + 2t_n^2 + 10t_n + 8)$

3. If the third order Taylor series method (two more terms than Euler's method) is used to solve $u' = -u^2$, write u_{n+1} in terms of h, t_n and u_n only. ($t_n = nh$, $u_n \approx u(t_n)$)

$$u'' = -2uu' = -2u(-u^2) = 2u^3$$

$$u''' = 6u^2u' = 6u^2(-u^2) = -6u^4$$

4

$$u_{n+1} = u_n + h(-u_n^2) + \frac{h^2}{2} 2u_n^3 + \frac{h^3}{6}(-6u_n^4)$$

4. Do **one** iteration of Newton's method, starting from $(1, 1)$, to solve:

$f(x, y) = x^2 + xy^3 - 9 = 0$

$g(x, y) = 3x^2y - y^3 - 4 = 0$

4

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} 2x + y^3 & 3xy^2 \\ 6xy & 3x^2 - 3y^2 \end{pmatrix}^{-1}_{(1,1)} \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ 6 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -7 \\ -2 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 3 \end{pmatrix}$$

5. a. A rootfinder produces consecutive root estimates of 2.1, 2.001, 2.000001, when the exact root is $r = 2$. Estimate the order of the method.

$.001 = M(.1)^\alpha$

$.000001 = M(.001)^\alpha$

$1000 = 100^\alpha$ $\quad \alpha = 1.5$

b. A differential equation approximation produces the approximation $u(1) = 2.001$ when $h = 0.01$ and $u(1) = 2.000001$ when $h = 0.001$. If the true solution is $u(1) = 2$, estimate the order of the method used.

6

$.001 = M(.01)^\alpha$

$.000001 = M(.001)^\alpha$

$1000 = 10^\alpha$ $\quad \alpha = 3$

6. How should A, r be chosen to make the approximation:

$\int_{-1}^{1} f(x)dx \approx Af(-r) + Af(0) + Af(r)$

as high degree of precision as possible?

4

$2 = \int_{-1}^{1} 1\,dx = A + A + A \quad A = \frac{2}{3}$

$0 = \int_{-1}^{1} x\,dx = A(-r) + A(0) + A(r) = 0 \quad r = \frac{1}{2} \quad r = \sqrt{\frac{1}{2}}$

$\frac{2}{3} = \int_{-1}^{1} x^2\,dx = Ar^2 + A0^2 + Ar^2 = 2Ar^2 = \frac{4}{3}r^2$

7. a. If $p_3(x)$ is the third degree (Lagrange) polynomial which satisfies $p_3(x_i) = f(x_i),\ i = 0, 1, 2, 3$, give a formula for the error $f(x) - p_3(x)$ at an arbitrary point x.

6

$\frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$

b. If $T_3(x)$ is the third degree (Taylor) polynomial which satisfies $T_3(x_0) = f(x_0), T_3'(x_0) = f'(x_0), T_3''(x_0) = f''(x_0), T_3'''(x_0) = f'''(x_0)$, give a formula for the error $f(x) - T_3(x)$ at an arbitrary point x.

$\frac{f^{(4)}(\xi)}{4!}(x-x_0)^4$

8. Will the iteration $x_{n+1} = \frac{1}{x_n - 1}$ converge to the root 1.618, if the starting guess is sufficiently good? Justify your answer.

4

$g(x) = \frac{1}{x-1} \quad g'(x) = \frac{-1}{(x-1)^2} \quad g'(1.618) = -2.618$

so no

9. Consider the linear system:

$$\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1+\epsilon \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

True or False:

a. If $\epsilon > 0$ the Jacobi iterative method for solving this system will converge to the true solution, no matter what starting point (x_0, y_0) is used.

True

2

b. If ϵ is close to 0, this matrix will have a large condition number.

True

Math 4329, Final

Name Key (j)

1. Use the power method to find the largest (in absolute value) eigenvalue of

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 8 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Start with $(1, 5, 1)$ and do 3 iterations. What is the corresponding eigenvector?

4

$$x_0 = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \quad x_1 = \begin{pmatrix} 6 \\ 42 \\ 6 \end{pmatrix} \quad x_2 = \begin{pmatrix} 48 \\ 348 \\ 48 \end{pmatrix} \quad x_3 = \begin{pmatrix} 396 \\ 2880 \\ 396 \end{pmatrix} \qquad Ax_2 \approx 8.25\, x_2$$

8.25

8.27

$$\lambda_{max} = 8.25 \qquad z_{max} = \begin{pmatrix} 0.1375 \\ 1 \\ 0.1375 \end{pmatrix}$$

2. a. If $p_4(x)$ is the fourth degree Lagrange polynomial which satisfies $p_4(x_i) = f(x_i)$, $i = 0, 1, 2, 3, 4$, give a formula for the error $f(x) - p_4(x)$ at an arbitrary point x.

2

$$f(x) - p_4(x) = \frac{f^{V}(\xi)}{5!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

b. If $T_4(x)$ is the fourth degree Taylor polynomial which satisfies $T_4(a) = f(a), T_4'(a) = f'(a), T_4''(a) = f''(a), T_4'''(a) = f'''(a), T_4^{iv}(a) = f^{iv}(a)$, give a formula for the error $f(x) - T_4(x)$ at an arbitrary point x.

2

$$f(x) - T_4(x) = \frac{f^{V}(\xi)}{5!}(x-a)^5$$

3. If the second order Taylor series method (one more term than Euler's method) is used to solve $u' = -tu^3$, write u_{n+1} in terms of h, t_n and u_n. ($t_n = nh, u_n \approx u(t_n)$)

$$u'' = -u^3 - t3u^2u' = -u^3 + 3t^2u^5$$

4

$$u_{n+1} = u_n + h(-t_n u_n^3) + \frac{h^2}{2}(-u_n^3 + 3t_n^2 u_n^5)$$

4. Do one iteration of Newton's method, starting from (0,0), to solve:

$f(x,y) = \sqrt{x+1} + xy + 3 = 0$
$g(x,y) = sin(x+2y) - ln(1+x) = 0$

4

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}(x+1)^{-\frac{1}{2}} + y & x \\ \cos(x+2y) - \frac{1}{1+x} & 2\cos(x+2y) \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$= -\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}$$

5. How should A, r be chosen to make the approximation:

$\int_{-1}^{1} f(x)dx \approx Af(-r) + Af(0) + Af(r)$

as high degree of precision as possible? What is the degree of precision then?

4

$f=1$ $\quad 2 = \int_{-1}^{1} 1\,dx = A + A + A$ $\quad A = \frac{2}{3}$

$f=x$ $\quad 0 = \int_{-1}^{1} x\,dx = -rA + Ar = 0$

$f=x^2$ $\quad \frac{2}{3} = \int_{-1}^{1} x^2 dx = Ar^2 + Ar^2 = \frac{4}{3}r^2$ $\quad r = \sqrt{\frac{1}{2}}$

$f=x^3$ $\quad 0 = \int_{-1}^{1} x^3 dx = -Ar^3 + Ar^3 = 0$ $\quad$ d.o.p $= 3$

2

$f=x^4$ $\quad \frac{2}{5} = \int_{-1}^{1} x^4 dx \neq \frac{2}{3}(\sqrt{\tfrac{1}{2}})^4 + \frac{2}{3}(\sqrt{\tfrac{1}{2}})^4 = \frac{1}{3}$

6. Consider the linear system:

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -3 \end{bmatrix}$$

a. Write out the equations for the Jacobi iterative method for solving this system (don't actually do any iterations).

2

$$x_{n+1} = (5 - 2y_n - z_n)/7$$
$$y_{n+1} = (4 - 2z_n)/3$$
$$z_{n+1} = (-3 - x_n + 3y_n)/5$$

b. Write out the equations for the Gauss-Seidel iterative method for solving this system.

2

Same as (a) but

$$z_{n+1} = (-3 - x_{n+1} + 3y_{n+1})/5$$

c. True or False: the Jacobi iterative method (6a) will converge for *any* starting vector (x_0, y_0, z_0). Give a reason for your answer.

1

True, A is diagonal dominant

d. Given that

2

$$A^{-1} = \begin{bmatrix} 0.1419 & -0.0878 & 0.0068 \\ 0.0135 & 0.2297 & -0.0946 \\ -0.0203 & 0.1554 & 0.1419 \end{bmatrix}$$

find the condition number of A (using L_∞ norm). If you were to solve the linear system above using Gaussian elimination with partial pivoting, would you expect serious roundoff errors?

$$\text{cond}(A) = \|A\|_\infty \|A^{-1}\|_\infty = 10\,(0.3378) = 3.378$$

NO

7. Will the iteration $x_{n+1} = \frac{3}{4}x_n + 1/x_n^3$ converge when x_0 is sufficiently close to the root $r = \sqrt{2}$? (Justify your answer theoretically, without actually iterating the formula.) If it converges, give the order of convergence.

4

$g(x) = \frac{3}{4}x + x^{-3}$

$g'(x) = \frac{3}{4} - 3x^{-4}$ $\quad g'(\sqrt{2}) = \frac{3}{4} - \frac{3}{(\sqrt{2})^4} = 0$

$g''(x) = 12x^{-5}$ $\quad g''(\sqrt{2}) \neq 0$ $\quad$ so converges, second order

8. a. Write the second order differential equation $u'' - 5u' - u = sin(t)$ as a system of two first order equations, that is, in the form:

2

$u' = f(t, u, v) = v$

$v' = g(t, u, v) = 5v + u + \sin(t)$

b. Now write out the formulas for u_{n+1}, v_{n+1} for Euler's method applied to this system of first order equations:

2

$u_{n+1} = u_n + h(v_n)$

$v_{n+1} = v_n + h(5v_n + u_n + \sin(t_n))$

9. a. A root finder gives consecutive errors of $e_8 = 10^{-3}, e_9 = 10^{-5}, e_{10} = 10^{-12}$. Estimate the order of the method.

2

$\alpha = 3.5$

$10^{-5} = M(10^{-3})^{\alpha}$

$10^{-12} = M(10^{-5})^{\alpha}$

$10^{7} = (10^{2})^{\alpha}$

b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-13} when $h = 10^{-4}$. Estimate the order of the method.

2

$10^{-5} = M(10^{-2})^{\alpha}$

$10^{-13} = M(10^{-4})^{\alpha}$

$10^{8} = (10^{2})^{\alpha}$ $\quad \alpha = 4$

c. A differential equation solver gives an answer $u(1) = 0.88888$ when $h = 0.1$, and $u(1) = 0.80666$ when $h = 0.01$, and $u(1) = 0.80600$ when $h = 0.001$. Estimate the order of the method.

$0.88888 - I = M(0.1)^{\alpha}$

$0.80666 - I = M(0.01)^{\alpha}$

$0.80600 - I = M(0.001)^{\alpha}$

2

$.08222 = M(0.1^{\alpha} - 0.01^{\alpha})$

$.00066 = M(0.01^{\alpha} - 0.001^{\alpha})$

$124.5 = 10^{\alpha}$ $\quad \alpha = 2.1$

Math 4329, Final (K)

Name Key

1. a. Let $T_3(x)$ be the Taylor polynomial of degree 3 which matches $f(x), f'(x), f''(x)$ and $f'''(x)$ at $x = 0$, where $f(x) = cos(4x)$. Find the best possible bound on

$max_{-2\le x\le 2}|T_3(x) - f(x)| \le$ $\left|\frac{f^{iv}(\xi)}{4!}(x-0)^4\right| \le \left|\frac{4^4 \cos(4\xi)}{24} 2^4\right| \le 170.\mathrm{Cl}$

3

b. Let $L_3(x)$ be the Lagrange polynomial of degree 3 which matches $f(x)$ at $x = -3, -2, 2$ and 3, where $f(x) = cos(4x)$. Find the best possible bound on

$max_{-2\le x\le 2}|L_3(x) - f(x)| \le$ $\left|\frac{4^4 \cos(4\xi)}{24}(x+3)(x+2)(x-2)(x-3)\right|$

3

$\le \frac{4^4}{24} 36 = 384$

assume $x=0$, $\xi=0$ worst case

2. a. A root finder gives consecutive errors of $e_8 = 10^{-5}, e_9 = 10^{-6}, e_{10} = 10^{-11}$. Estimate the order of the method.

2

$10^{-6} = M(10^{-5})^\alpha$ $\quad 10^5 = 10^\alpha \quad \alpha = 5$

$10^{-11} = M(10^{-6})^\alpha$

b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-13} when $h = 10^{-4}$. Estimate the order of the method.

2

$\frac{10^{-5} = M(10^{-2})^\alpha}{10^{-13} = M(10^{-4})^\alpha}$ $\quad 10^8 = (10^2)^\alpha \quad \alpha = 4$

3. Use the inverse power method to determine the smallest eigenvalue of

$$A = \begin{bmatrix} \frac{-4}{6} & \frac{2}{6} \\ \frac{5}{6} & \frac{-1}{6} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

4

Start the iteration with $(x_0, y_0) = (3, 8)$, and do 2 iterations.

$$V_0 = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad V_1 = \begin{pmatrix} 19 \\ 47 \end{pmatrix} \quad V_2 = \begin{pmatrix} 113 \\ 283 \end{pmatrix}$$

5.95

6.02

$\lambda_{max}(A^{-1}) \approx 6$

$\lambda_{min}(A) = \frac{1}{6}$

4. $x^2 + xy^3 + 3y = 9$
$3x^2y - y^3 - 2x = 4$

4

Do one iteration of Newton's method, to find a root of this system, starting from $(x_0, y_0) = (0, 0)$.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} 2x + y^3 & 3xy^2 + 3 \\ 6xy - 2 & 3x^2 - 3y^2 \end{pmatrix}^{-1}_{(0,0)} \begin{pmatrix} -9 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -9 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

5. Take one step of a second order Taylor series method (Euler is the first order Taylor method) with $h = 0.001$ to approximate the solution of the following problem, at $t = 0.001$:

$u' = 4t + u^3$
$u(0) = 2$

4

$$u' = 4t + u^3 \qquad u'(0) = 4(0) + 2^3 = 8$$

$$u'' = 4 + 3u^2u' \qquad u''(0) = 4 + 3(2)^2(8) = 100$$

$$u(h) \approx u(0) + u'(0)h + u''(0)\frac{h^2}{2} = 2 + 8h + \frac{100}{2}h^2$$

≈ 2.008050

6. Will the iteration $x_{n+1} = 2 - \frac{3}{2}x_n + \frac{1}{2}x_n^3$ converge when x_0 is sufficiently close to the root $r = 1$? If so, what is the order of convergence? (Justify your answer theoretically, without actually iterating the formula.)

$g(x) = 2 - \frac{3}{2}x + \frac{1}{2}x^3$ $\qquad g(1) = 1$

$g'(x) = -\frac{3}{2} + \frac{3}{2}x^2$ $\qquad g'(1) = 0$ $\qquad$ converges quadratically

$g''(x) = 3x$ $\qquad g''(1) = 3$

4

7. a. Reduce

$$y'' = 3y'y - e^t z'$$
$$z'' = z'z - \sqrt{y}$$

to a system of 4 first order equations. The right hand sides must involve only $t, u1, u2, u3, u4$.

$u1 = y$
$u2 = y'$
$u3 = z$
$u4 = z'$

$u1' = u2$

$u2' = 3(u2)(u1) - e^t(u4)$

$u3' = u4$

$u4' = (u4)(u3) - \sqrt{u1}$

2

b. Now write out the formulas for $u1_{n+1}, u2_{n+1}, u3_{n+1}, u4_{n+1}$ for Euler's method applied to this system of first order equations:

$u1_{n+1} = u1_n + h\, u2_n$

$u2_{n+1} = u2_n + h(3\, u2_n(u1_n) - e^{t_n} u4_n)$

$u3_{n+1} = u3_n + h\, u4_n$

$u4_{n+1} = u4_n + h\,(u4_n(u3_n) - \sqrt{u1_n})$

2

8. Write $\frac{(8+x)^{\frac{1}{3}}-2}{x}$ in a form where there is no serious problem with round-off, when $x \approx 0$. (Hint: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.)

$a = (8+x)^{\frac{1}{3}}$
$b = 2$

4

$$\frac{(8+x)-8}{x\left[(8+x)^{\frac{2}{3}}+2(8+x)^{\frac{1}{3}}+4\right]} = \frac{1}{4+2(8+x)^{\frac{1}{3}}+(8+x)^{\frac{2}{3}}}$$

9. Consider the linear system:

$$\begin{bmatrix} 1 & 2.0000001 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

a. Write out the equations for the Jacobi iterative method for solving this system (don't actually do any iterations).

2

$$x_{n+1} = 5 - 2.0000001\, y^n$$
$$y_{n+1} = 1 - 0.5 x_n$$

b. Write out the equations for the Gauss-Seidel iterative method for solving this system.

2

$$x_{n+1} = 5 - 2.0000001\, y_n$$
$$y_{n+1} = 1 - 0.5\, x_{n+1}$$

c. Calculate the condition number for this matrix. Hint, if:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / (ad - bc)$$

2

If machine precision is $\epsilon = 10^{-16}$, about how many significant figures would you expect in the solution, if Gauss elimination with partial pivoting is used to solve this linear system?

$$A^{-1} = \frac{\begin{bmatrix} 4 & -2.0000001 \\ -2 & 1 \end{bmatrix}}{-0.0000002} \approx \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot (-5 \cdot 10^6)$$

$$\text{Cond}(A) = \|A\|\|A^{-1}\| = 6 \cdot 6 \cdot 5 \cdot 10^6 \approx 1.8 \cdot 10^8$$

about 8 digits

Math 4329, Final (2)

Name Key

1. If the second order Taylor series method (one more term than Euler's method) is used to solve $u' = t^2(1+u^2)$, write u_{n+1} in terms of h, t_n and u_n only. ($t_n = nh, u_n \approx u(t_n)$)

$$u'' = t^2 2uu' + 2t(1+u^2)$$
$$= 2t^2u(t^2)(1+u^2) + 2t(1+u^2)$$
$$= (2t + 2t^4u)(1+u^2)$$

4

$$u_{n+1} = u_n + h\, t_n^2(1+u_n^2) + \frac{h^2}{2}(2t_n + 2t_n^4 u_n)(1+u_n^2)$$

2. a. Let $T_4(x)$ be the Taylor polynomial of degree 4 which matches $f(x), f'(x), f''(x), f'''(x)$ and $f^{iv}(x)$ at $a = -0.1$, where $f(x) = x^6 + x^3$. Use the Taylor remainder formula to find a reasonable bound on

6

$$|T_4(0) - f(0)| \le \frac{f^{V}(x)}{5!}(0-(-.1))^5 = \frac{720(x)}{120}10^{-5} = 6(0.1)10^{-5}$$
$$= 6 \cdot 10^{-6}$$

b. Let $L_4(x)$ be the Lagrange polynomial of degree 4 which matches $f(x)$ at $x = -0.1, 0.1, 0.2, 0.3$ and 0.4, where $f(x) = x^6 + x^3$. Use the Lagrange error formula to find a reasonable bound on

$$|L_4(0) - f(0)| \le \left|\frac{f^{V}(x)}{5!}(0+0.1)(0-0.1)(0-0.2)(0-0.3)(0-0.4)\right|$$

$$\le \left|\frac{720(x)}{120} 24 \cdot 10^{-5}\right| \le 6(0.4)24 \cdot 10^{-5} = 5.76 \cdot 10^{-4}$$

3. a. A rootfinder produces consecutive errors of $0.01, 0.0003, 0.000001$. Estimate the order of the method.

$$\frac{.0003 = M(0.01)^\alpha}{.000001 = M(0.0003)^\alpha} \qquad 300 = (33.\bar{3})^\alpha$$

$$\alpha = 1.63$$

4

b. A quadrature method produces estimates of an integral of 5.51 when $h = 0.1$ and 5.50007, when $h = 0.01$, and the exact integral is 5.5. Estimate the order of the method.

$$\frac{.01 = M(0.1)^\alpha}{.00007 = M(0.01)^\alpha} \qquad 143 = 10^\alpha \qquad \alpha = 2.15$$

4. If

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix},$$

9

a. Calculate the condition number of A.

$$\|A\|_\infty \|A^{-1}\|_\infty = 3 \cdot 5 = 15$$

b. Estimate the smallest (in absolute value) eigenvalue of A, and the corresponding eigenvector, using the inverse power iteration. Start with $x_0 =< 1, 1, 1, 1 >$ and do 4 iterations.

$$x_1 = A^{-1}x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} -1 \\ 1 \\ -2 \\ 2 \end{pmatrix} \quad x_3 = \begin{pmatrix} 4 \\ -3 \\ 8 \\ -6 \end{pmatrix} \quad x_4 = \begin{pmatrix} -13 \\ 10 \\ -27 \\ 21 \end{pmatrix}$$

$$A^{-1}x_3 = x_4 \approx -3.3x_3 \qquad \text{so} \quad -\frac{1}{3.3}\vec{x}_3 \cong A\vec{x}_3$$

$$\lambda_{min} \approx -0.3 \qquad \vec{x}_{min} \approx \begin{pmatrix} -13 \\ 10 \\ -27 \\ 21 \end{pmatrix}$$

c. Do one iteration of Newton's method, starting from $(0,0,0,0)$ to solve:

$f1(x1,x2,x3,x4) = x2 + x3 + x4 = 0$
$f2(x1,x2,x3,x4) = x1 + x2 - 1 = 0$
$f3(x1,x2,x3,x4) = x1 + x4 = 0$
$f4(x1,x2,x3,x4) = x3 + x4 = 0$
(Hint: notice that the Jacobian matrix is just A.)

$$\vec{X}_1 = \vec{X}_0 - A^{-1}\vec{f} = -\begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

5. How should A, B, r be chosen to make the approximation:

$\int_{-1}^{1} f(x)dx \approx Af(-r) + Bf(0) + Af(r)$

as high degree of precision as possible?

4

$$2 = \int_{-1}^{1} 1\,dx = A + B + A$$
$$0 = \int_{-1}^{1} x\,dx = 0$$
$$\frac{2}{3} = \int_{-1}^{1} x^2\,dx = A(-r)^2 + B(0)^2 + A(r^2) = 2Ar^2$$
$$0 = \int_{-1}^{1} x^3\,dx = 0$$
$$\frac{2}{5} = \int_{-1}^{1} x^4\,dx = A(-r)^4 + B(0)^4 + A(r)^4 = 2Ar^4$$
$$r^2 = \frac{2Ar^4}{2Ar^2} = \frac{2/5}{2/3} = \frac{3}{5} \quad r = \sqrt{0.6}$$

$$A = \frac{2/3}{2r^2} = \frac{1}{3r^2} = \frac{5}{9} = A$$
$$B = 2 - 2A = \frac{8}{9} = B$$

3

6. Write $\frac{\sqrt{4+x}-2}{x}$ in a form where there is no serious problem with roundoff, when $x \approx 0$.

$$\frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = \frac{1}{\sqrt{4+x}+2}$$

6

7. a. Write the third order differential equation $u''' - 3u'' - u^3 = e^t$ as a system of three first order equations, that is, in the form:

$$u' = f(t,u,v,w) = v$$
$$v' = g(t,u,v,w) = w$$
$$w' = h(t,u,v,w) = 3w + u^3 + e^t$$

b. Now write out the formulas for $u_{n+1}, v_{n+1}, w_{n+1}$ for Euler's method applied to this system of first order equations:

$$u_{n+1} = u_n + h\, v_n$$
$$v_{n+1} = v_n + h\, w_n$$
$$w_{n+1} = w_n + h(3w_n + u_n^3 + e^{t_n})$$

4

8. Will the iteration $x_{n+1} = 4\, x_n(1 - x_n)$ converge to the root 0.75, if the starting guess is sufficiently good? **Justify** your answer.

$$g(x) = 4x(1-x) = 4x - 4x^2$$
$$g'(x) = 4 - 8x \qquad g'\left(\tfrac{3}{4}\right) = -2 \quad \text{so} \quad \text{no}$$

Math 4329, Final (0)

Name Key

1. a. Let $T_4(x)$ be the Taylor polynomial of degree 4 which matches $f(x), f'(x), f''(x), f'''(x)$ and $f^{iv}(x)$ at $x = 1$, where $f(x) = e^{-2x}$. Find a reasonable bound on

$$|T_4(1.2) - f(1.2)| \leq \left|\frac{f^{v}(c)\,(x-1)^5}{5!}\right| = \left|\frac{32e^{-2c}(0.2)^5}{120}\right|$$

$$\leq \frac{32e^{-2}}{120}(0.2)^5 = 1.15\cdot 10^{-5}$$

$1 \leq c \leq 1.2$

b. Let $L_4(x)$ be the Lagrange polynomial of degree 4 which matches $f(x)$ at $x = 0, 0.5, 1, 1.5$ and 2, where $f(x) = e^{-2x}$. Find a reasonable bound on

$$|L_4(1.2) - f(1.2)| \leq \left|\frac{f^{v}(c)}{5!}(1.2-0)(1.2-0.5)(1.2-1)(1.2-1.5)(1.2-2)\right|$$

$$\leq \left|\frac{32e^{-2c}}{120}.0040\right| = \frac{32}{120}(0.0040) = 0.0107$$

$0 \leq c \leq 2$

2. If $a = 0, b = 1000$ and $f(a)$ and $f(b)$ have opposite signs, how many bisection iterations are required to find a root between a and b to an accuracy of 10^{-8}?

$$\frac{1000}{2^n} = 10^{-8} \qquad 2^n = 10^{11}$$

$$n = \frac{\ln 10^{11}}{\ln 2} \approx 37$$

3. Reduce

$$y'' = 3y'y - e^t z$$
$$z'' = z'z - \sqrt{y}$$

$u_1 = y$
$u_2 = y'$
$u_3 = z$
$u_4 = z'$

to a system of 4 first order equations. The right hand sides must involve only t, u_1, u_2, u_3, u_4. The left hand sides must be u_1', u_2', u_3', u_4' respectively.

3

$u_1' = u_2$
$u_2' = 3u_2u_1 - e^t u_3$
$u_3' = u_4$
$u_4' = u_4 u_3 - \sqrt{u_1}$

4. a. A root finder gives consecutive errors of $e_8 = 10^{-3}, e_9 = 10^{-5}, e_{10} = 10^{-12}$. Estimate the order of the method.

2

$10^{-5} = M(10^{-3})^\alpha$
$10^{-12} = M(10^{-5})^\alpha$

$10^7 = (10^2)^\alpha$ $\quad \alpha = 3.5$

b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-12} when $h = 10^{-4}$. Estimate the order of the method.

2

$10^{-5} = M(10^{-2})^\alpha$
$10^{-12} = M(10^{-4})^\alpha$

$10^7 = (10^2)^\alpha$ $\quad \alpha = 3.5$

c. A differential equation solver gives an answer $u(1) = 1.08888$ when $h = 0.1$, and $u(1) = 1.00666$ when $h = 0.01$, and $u(1) = 1.00600$ when $h = 0.001$. Estimate the order of the method.

2

$1.08888 - I = M(.1)^\alpha$
$1.00666 - I = M(.01)^\alpha$
$1.00600 - I = M(.001)^\alpha$

$$\frac{.08222}{.00066} = \frac{M(.1)^\alpha - M(.01)^\alpha}{M(.01)^\alpha - M(.001)^\alpha}$$

$$125 = \frac{(.1)^\alpha - (.01)^\alpha}{(.01)^\alpha - (.001)^\alpha} = 10^\alpha$$

$\alpha = 2.1$

5. Find A, B which make the approximation

$\int_0^h f(x)dx \approx Ahf(0.2h) + Bhf(0.9h)$

as high order as possible.

4

$A + B = 1$

$0.2A + 0.9B = 0.5$

$h = \int_0^h 1\, dx = Ah + Bh$

$\frac{h^2}{2} = \int_0^h x\, dx = Ah(0.2h) + Bh(0.9h)$

$A = \frac{4}{7}$

$B = \frac{3}{7}$

6. Use the inverse power method to determine the smallest eigenvalue of

$$A = \begin{bmatrix} \frac{4}{6} & \frac{-2}{6} \\ \frac{-5}{6} & \frac{1}{6} \end{bmatrix}$$

$A^{-1} = \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix}$

4

Start the iteration with $(x_0, y_0) = (3, 8)$.

-6.01

$z_0 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$ $z_1 = \begin{pmatrix} -19 \\ -47 \end{pmatrix}$ $z_2 = \begin{pmatrix} 113 \\ 283 \end{pmatrix}$ $z_3 = \begin{pmatrix} -679 \\ -1697 \end{pmatrix}$

-6.00

$A^{-1} z_2 = -6 z_2$

$-\frac{1}{6} z_2 = A z_2$

$\lambda = -\frac{1}{6}$

7. Will the iteration $x_{n+1} = 2 - \frac{3}{2}x_n + \frac{1}{2}x_n^3$ converge when x_0 is sufficiently close to the root $r = 1$? If so, what is the order of convergence? (Justify your answer theoretically, without actually iterating the formula.)

4

$g(x) = 2 - \frac{3}{2}x + \frac{1}{2}x^3$

$g'(x) = -\frac{3}{2} + \frac{3}{2}x^2$

$g''(x) = 3x$

$g'(r) = 0$

$g''(r) = 3$

Converges quadratically

8. If the second order Taylor series method (one more term than Euler's method) is used to solve $u' = t^2\sqrt{1+u^2}$, write u_{n+1} in terms of h, t_n and u_n only. ($t_n = nh, u_n \approx u(t_n)$)

$$u'' = \frac{t^2}{2}(1+u^2)^{-\frac{1}{2}} 2uu' + 2t(1+u^2)^{\frac{1}{2}} = t^2 u t^2 + 2t\sqrt{1+u^2}$$

4 $$u_{n+1} = u(t_n+h) = u_n + h t_n^2\sqrt{1+u_n^2} + \frac{h^2}{2}\left[t_n^4 u_n + 2t_n\sqrt{1+u_n^2}\right]$$

9. Do **two** iterations of Newton's method, starting from $(0,0)$, to solve:

$f(x,y) = 2x + y - 4 = 0$
$g(x,y) = -6x + y + 4 = 0$

4 $$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -6 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -4 \\ 4 \end{pmatrix} = \frac{\begin{pmatrix} 1 & -1 \\ 6 & 2 \end{pmatrix}}{8}\begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -6 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

10. Will the following iteration converge (to something)? **Justify** your answer without actually iterating.

4

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.75 \\ -0.75 & 0.5 \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$\det\begin{pmatrix} 0{,}5-\lambda & 0{,}75 \\ -0{,}75 & 0{,}5-\lambda \end{pmatrix} = (0{,}5-\lambda)^2 + 0{,}75^2$$

$$\lambda = 0{,}5 \pm 0{,}75 i$$

$$|\lambda| = \sqrt{0{,}5^2 + (0{,}75)^2} = 0{,}901$$

so converges