

a reasonable bound on
$$f(k) = \frac{25m}{(0r^3)}$$

$$|f(x)-t_2(x)| = \left|\frac{f''(y)}{3!}x^3\right| = \left|\frac{2\sin y}{6\cos^3 y}(0.1)^3\right| \leq \frac{2(\sin 0.1)}{6\cos^3 y}(0.1)^3$$

2. A certain computer stores floating point numbers in a 128-bit word. If a floating point number is written in normalized binary form $(1.xxxxx..._2*$ 2e), it is stored using one sign bit (0 if the number is positive), then e+4095 is stored in binary in the next 13 bits, and then the mantissa xxxxx... is stored in the final 114 bits. Show exactly how the number -12.25 would be stored on this computer. Also: approximately how many decimal digits of accuracy does this machine have?

3. Compute the experimental order of convergence for a root finder with errors in 3 consecutive iterations of 10^{-5} , 10^{-7} and 10^{-14} .

4. The fixed point iteration $x_{n+1} = x_n + \sin(x_n)$ has roots at $r = n\pi$ for any integer n. Will this iteration converge if you start very close to the root r = 0? Will it converge if you start near the root $r = \pi$? In both cases, if it does converge, what is the order of convergence?

$$g(x) = x + stax$$

$$g'(x) = 1 + cax$$

$$g''(x) = -sin x$$

$$g'''(x) = -corx$$

$$f'''(x) = -corx$$

5. Show how Newton's method could be used to find $b^{\frac{1}{m}}$ for b > 0, where m is a positive integer, without doing anything other than add, subtract, multiply and divide. $f(x) = x^m - k = 0$

$$X_{nr_1} = X_n - \frac{X_n^{n-1}}{m X_n^{n+1}} = (1 - \frac{1}{m}) X_n + \frac{1}{m X_n^{n-1}}$$

6. Write the secant iteration for solving f(x) = 1/x - b = 0, in a form where no divisions are required (thus this iteration could be used to compute 1/b on a computer which cannot do divisions).

$$X_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - b\right)}{\left(\frac{1}{x_n} - b\right) - \left(\frac{1}{x_{n-1}} - b\right)}$$

$$X_{n+1} = x_n + x_{n+1} - b \times_n x_{n-1}$$

$$X_{n+1} = x_n + x_{n+1} - b \times_n x_{n-1}$$

Math 4329, Test I (\mathcal{F})

a. If $f(x) = x^5 + 2x^2$, find the Taylor polynomial $T_3(x)$ of degree 3

which matches
$$f$$
, f' , f'' and f''' at $a = 1$.

$$f(t) = 3$$

$$f(t) = 4$$

$$f(t) = 5$$

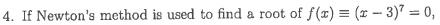
b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_3(x)|$ in the interval $-0.5 \le x \le 1.5$.

$$|f(x)-f_3(x)| = \left|\frac{f''(\epsilon)}{4!}(x-1)^4\right| = \left|\frac{120\epsilon}{4!}(x-1)^4\right| = \frac{120(1.5)}{24}(0.5)^4$$

$$\left(\text{or } 25.31\right) = 37.97$$

2. Write the quadratic formula root $[-b+\sqrt{b^2-4ac}]/(2a)$ in a form so that there are no serious problems with roundoff error, when b is positive and very large compared to ac.

3. Write out (and simplify) a secant method iteration, used to find \sqrt{b} , which does only basic arithmetic (add, subtract, mutliply and divide, no square roots). f(x) = x2-b=0



a. Will Newton's method converge for x_0 close to the root r = 3? $e_{1+1} = \frac{6}{7}e_{1}$. Explain.

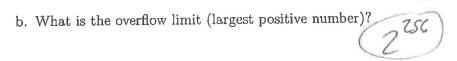
$$x_{n+1} = x_n - \frac{(x_n - 3)^7}{7(x_n - 3)^6} = x_n - \frac{1}{7}(x_n - 3)$$

b. What is the order of convergence, if it converges?

c. Will Newton's method converge for all x_0 ? Explain.

$$(e_n = \left(\frac{6}{7}\right)^n e_0 \rightarrow 0) \text{ any } x_0$$

- 5. A certain computer stores floating point numbers in a 32-bit word, which includes 1 sign bit, 9 bits for the exponent, and 22 bits for the mantissa. Approximately
 - a. What is the underflow limit (smallest positive number)?



- c. What is the machine precision (smallest positive number such that $1+\epsilon>1$)?
- 6. If the fixed point iteration $x_{n+1} = x_n + cf(x_n)$ is used near a root r of f(x) = 0, how should the constant c be chosen in order to ensure the fastest convergence?

$$g(x) = x + cf(x)$$
 $g'(x) = 1 + cf(x) = 0$
 $g'(x) = 1 + cf(x)$ $c = -1$
 $g'(x) = 1 + cf(x)$



Name Key

1. a. If $f(x) = 2x^4 + x^3$, find the Taylor polynomial $T_3(x)$ of degree 3

a. If
$$f(x) = 2x^3 + x^3$$
, that the Taylor polyholmal $f_3(x)$ of degree 3

which matches f, f', f'' and f''' at $a = 1$.

$$f(x) = 2x^3 + x^3$$
which matches f, f', f'' and f''' at $a = 1$.

$$f' = 8x^2 + 3x^3$$

$$f'' = 24x^2 + 6x^3$$

FN = 48 b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_3(x)|$ in the interval $-1 \le x \le 3$.

2. Write the quadratic formula root $[-b+\sqrt{b^2-4ac}]/(2a)$ in a form so that there are no serious problems with roundoff error, when b is positive and very large compared to ac.

3. If Newton's method is used to find a root of $f(x) \equiv (x-3)^5 = 0$, for what range of starting values x_0 will we get convergence to the root r=3? What is the order of convergence of Newton's method in this case?

$$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)} = X_n - \frac{(x_n-3)^3}{5(x_n-3)^4} = X_n - \frac{f(x_n-3)^3}{5(x_n-3)^4} = X_$$

4. If a = -10, b = 10 and f(a) and f(b) have opposite signs, about how many bisection iterations are required to find a root between a and b to an accuracy of 10^{-10} ?

$$\frac{ba}{2^{N}} = 10^{-10} \qquad 2^{N} = \frac{20}{10^{-10}} = 2.10^{11}$$

$$N \approx \frac{b(2.10^{11})}{2} = 38$$

5. Compute the experimental order of convergence for a root finder with errors in 3 consecutive iterations of 10^{-4} , 10^{-7} and 10^{-14} .

$$10^{-7} = M(10^{-4})^{10}$$
 $10^{-14} = M(10^{-7})^{10}$
 $(0^{-7})^{10} = (10^{3})^{10}$
 $3x = 7$

6. The fixed point iteration $x_{n+1} = x_n + \sin(x_n)$ has roots at $r = n\pi$ for any integer n. For which of these roots (which values of n) will the interation converge? What will be the order of convergence for these roots?

$$g(x_1) = x_1 + SM(x_1)$$

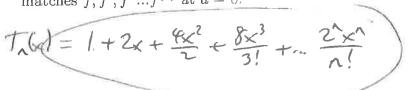
 $g'(x) = 1 + Cer(x)$ (nod): $g'(n\pi) = 0$ conv
 $g''(x) = -sM(x)$ (not) = 2 diverging
 $g'''(x) = -cor(x)$

$$\int_{0}^{\infty} \int_{0}^{\infty} \int_{0$$

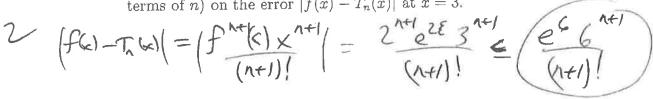
Math 4329, Test I (k)

Name Key

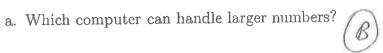
1. a. If $f(x) = e^{2x}$, find the Taylor polynomial $T_n(x)$ of degree n which matches $f, f', f''...f^{(n)}$ at a = 0.



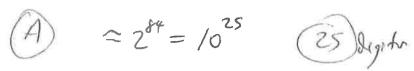
b. Use the Taylor remainder formula to get a reasonable bound (in terms of n) on the error $|f(x) - T_n(x)|$ at x = 3.



2. Computer A stores floating point numbers in a 96-bit word, which includes 1 sign bit, 11 bits for the exponent, and 84 bits for the mantissa. Computer B stores floating point numbers in a 96-bit word, with 1 sign bit, 25 bits for the exponent, and 70 bits for the mantissa.



b. Which computer has higher accuracy? Approximately how many significant decimal digits of accuracy does this computer have?





3. A root-finder produces approximations $x_3 = 5.01, x_4 = 5.0001, x_5 =$ 5.000 000 06 where one root is r=5? Estimate the experimental order of convergence. What method have we studied that has approximately this order?

1666 = 100× 12 = 1,61

4. Write $\frac{\sqrt{4+x-2}}{x}$ in a form where there is no serious problem with roundoff, when $x \approx 0$. 54x -2 54x +2 = (4+x)-4 × 54x +2 = x (54x+2)=

a. Newton's method is sometimes used to find
$$\frac{1}{b}$$
 by computing the root of $f(x) = b - \frac{1}{b}$. Write the Newton iteration in a form where



root of $f(x) = b - \frac{1}{x}$. Write the Newton iteration in a form where no divisions are required (thus we can find $\frac{1}{b}$ without doing any divisions).

b. Same as (5a) but use the secant method.

Same as (sa) but use the secant method.

$$\begin{array}{c}
X_{n+1} = X_n - (b - \frac{1}{X_n}) - (b - \frac{1}{X_n}) \\
X_n - X_{n-1}
\end{array}$$

6. The polynomial $x^3 - x^2 - x - 1$ has one real root, at x = 1.839. We can write $x^3 - x^2 - x - 1 = 0$ in the form $x^3 = x^2 + x + 1$, or $x = 1 + \frac{1}{x} + \frac{1}{x^2}$ and try the iteration $x_{n+1} = 1 + \frac{1}{x_n} + \frac{1}{x_n^2}$.

Will this converge, for x_0 near 1.839? Justify your answer without actually doing any iterations.

3

$$g(6) = -\frac{1}{2} - \frac{2}{23}$$

$$g'(1.839) = -1$$
 $(1.839)^2$ $(1.839)^3$

$$\left(\mathcal{T}_{2} (x) = x - \frac{x^{2}}{2} \right)$$

$$f(0)=0$$
 $f(x) = h(1+sin \times)$
 $f'(0)=1$ $f' = \frac{1}{1+sin \times}$
 $f''(0)=-1$ $f'' = \frac{1}{1+sin \times}$
 $f''' = \frac{1}{1+sin \times}$

050501

b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_2(x)|$ at x = 0.1.

$$\left(f(x) - f(x)\right) = \left|\frac{f'''(c)}{3!} \times^3\right| = \left(\frac{cor(c)}{(1+sinc)^2} \frac{(O-1)^3}{6}\right) = \left(\frac{coo(c)}{6}\right) = 1.66.1c$$

2. A certain computer stores floating point numbers in a 96-bit word. If a floating point number is written in normalized binary form (1.xxxxx...₂* 2^e), it is stored using one sign bit (0 if the number is positive), then e+2047 is stored in binary in the next 12 bits, and then the mantissa xxxxx... is stored in the final 83 bits. Show exactly how the number -17.125 would be stored on this computer. Also, approximately how many **decimal** digits of accuracy does this machine have?

$$-17.125 = -10.0.01.001_{2} = -1.0001001.2^{4}$$

$$1.100000000001100010010000... 0000$$

$$7047+4$$

$$-83$$

$$-83$$

$$-10^{-25}$$

$$2047+4$$
 = 2051 1

(25 dights)

(30)

C01890.0

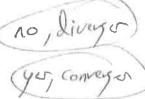


3

3. The polynomial
$$x^2-x-1$$
 has two roots, $r_1=-0.618, r_2=1.618$. We can write $x^2-x-1=0$ in the form $x^2=x+1$, and then $x=1+\frac{1}{x}$ and try the iteration $x_{n+1}=1+\frac{1}{x_n}$.

Will this converge, for x_0 near $r_1 = -0.618$? Justify your answer theoretically. Will it converge near $r_2 = 1.618$?

$$g(x) = 1 + \frac{1}{x}$$
 $g'(-0.618) = -2.62$ (no, diveyor)
 $g'(x) = -\frac{1}{x}$ $g'(1.618) = 0.382$ (yer, conveyor)



4. Estimate the experimental order of convergence for a root finder with errors in 3 consecutive iterations of $3 * 10^{-3}$, $2 * 10^{-4}$ and $5 * 10^{-8}$.

$$4000 = 15^{d}$$
 $d = \frac{h4000}{als} = 3.06$

a. $r = \sqrt{a}$ is a root of $f(x) = x^2 - a = 0$. Write Newton's iteration for finding this root.

$$2 \qquad \times_{\Lambda q} = \times_{\Lambda} - \frac{f(x_1)}{f(x_1)} = \times_{\Lambda} - \frac{x_1^2 - x_2}{2x_{\Lambda}} = \left(\frac{1}{2}(x_1 + \frac{q}{x_1})\right)$$

b. Given that, for Newton's method:

$$x_{n+1} - r = \frac{f''(c_n)}{2f'(x_n)}(x_n - r)^2$$

where c_n is between x_n and the root r, show that the iteration in 5a will converge provided $x_0 > \sqrt{a}/3$. (Hint: just show that x_1 will be closer to the root than x_0 .)

should be a I in abr. value

Math 4329, Test I



 $f' = 4x^{3} - 6x^{2} + 1$ $f'' = 12x^{2} - 12x$ f''' = 24x - 12 f''' = 24

a. If $f(x) = x^4 - 2x^3 + x + 3$, find the Taylor polynomial $T_3(x)$ of degree 3 which matches f, f', f'' and f''' at a = 1.

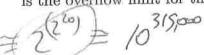
$$T_3(x) = f(1) + f'(1)(x-1) + f''(1)(x-1)^2 + f''(1)(x-1)^2$$

$$= 3 + (-1)(x-1) + 2(x-1)^3$$

b. Use the Taylor remainder formula to get a reasonable bound (in terms of x) on the error $|f(x) - T_3(x)|$ at x.

2. Computer A stores floating point numbers in a 128-bit word, which includes 1 sign bit, 21 bits for the exponent, and 106 bits for the mantissa. Computer B stores floating point numbers in a 128-bit word, with 1 sign bit, 13 bits for the exponent, and 114 bits for the mantissa.

a. Which computer can handle larger numbers? Approximately what is the overflow limit for this computer?



b. Which computer has higher accuracy? Approximately how many significant decimal digits of accuracy does this computer have?

$$B) \quad 2^{-1/4} = 5.10^{-35}$$

$$(35)$$

3. Compute the experimental order of convergence for a root finder with errors in 3 consecutive iterations of 0.03, 0.002 and 0.00001.

4. Will the iteration $x_{n+1} = 4x_n(1-x_n)$ converge when x_0 is sufficiently close to the root $r = \frac{3}{4}$? (Justify your answer theoretically, without actually iterating the formula.) If it converges, what is the order?

$$g(x) = 4x - 4x^{2}$$

 $g'(x) = 4 - 8x$ (10)
 $g'(\frac{3}{4}) = -2$

5. Show how Newton's method could be used to find $b^{p/q}$, where p, q are integers, and b > 0, without doing anything other than add, subtract, multiply and divide.

$$f(x) = x^2 - b^2 = 0$$

 $x_{nq} = x_n - \frac{x_n^2 - b^2}{2^2 x_n^2 - 1} = (1 - \frac{1}{2})x_n + \frac{b^2}{2^2 x_n^2 - 1}$

6. For the secant method, $e_{n+1} \approx Me_ne_{n-1}$. If the order of the secant method is α (ie, $e_{n+1} \approx Ce_n^{\alpha}$), show that α must satisfy $\alpha = 1 + 1/\alpha$.

$$\frac{e_n = ce_{n-1}^{\times}}{(e_n)^{\frac{1}{2}} = e_{n-1}} = e_n = e$$

Math 4329, Test I (K)

Name Key

1. a. If $f(x) = e^{x/2}$ use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_n(x)|$ for $-2 \le x \le 2$, where $T_n(x)$ is the Taylor polynomial of degree n for f(x), at a = 0.

(f n+1)! x n+1 < (2) n+1 est 2 n+1 & e' (n+1)!

b. Approximately how large does n need to be so that this error bound is less than 10^{-10} ?

 $\frac{Q}{(n+1)!} = 10^{-6}$ $(n+1)! = 2.7.10^{6} \Rightarrow (n \Rightarrow 13)$

- 2. A certain computer stores floating point numbers in a 128-bit word, which includes 1 sign bit, 17 bits for the exponent, and 110 bits for the mantissa (significand). Assuming a normalized binary form is used $(1.xxxxx...2*2^e)$ approximately what are:
 - a. the overflow limit (largest positive number)

2(216) = 265536 = 1019728

b. the machine precision (smallest $\epsilon > 0$ such that $1 + \epsilon > 1$)

 $2^{-1/0} = 7.7.10^{-34}$

- 3. Consider that fixed-point iteration $x_{n+1} = 2.5x_n(1-x_n)$.
 - a. What are the two roots (points r such that if $x_n = r$, x_{n+1} will still equal r)?

2

b. Analyze each root to determine if the iteration will converge (and if so, with what order) when you start close to that root.

$$g(x) = 2.5 \times -2.5 \times^{2} \qquad g'(0) = 2.5 \qquad \text{diverger } r = 0$$

$$g'(x) = 2.5 - 5 \times \qquad g'(0.6) = -0.5 \qquad \text{converger } r = 0.6$$

$$\text{linearly}$$

4. Estimate the experimental order of convergence for a root finder with errors in 3 consecutive iterations of 10^{-5} , 10^{-7} and 10^{-14} .

$$10^{-7} = M(10^{-5})^{4}$$
 $10^{-4} = M(10^{-7})^{4}$
 $16^{7} = 10^{24}$
 $(4 = 3.5)$

- 5. The root of $f(x) \equiv \frac{1}{x} b = 0$ is $x = \frac{1}{b}$.
 - a. Write Newton's iteration for solving f(x) = 0 in a form so that no divisions are required; thus providing a way to find $\frac{1}{b}$ without doing any divisions.

$$X_{nej} = x_n - \frac{x_1 - b}{(x_1)} = x_n + (\frac{1}{x_n} - b)x_n^2 = (x_n (2 - bx_n))$$

b. Same problem, but use the secant iteration.

$$\sum_{x_{n-1}} x_{n-1} = x_{n} - \left(\frac{1}{x_{n}} - b\right) = x_{n} + \left(\frac{1}{x_{n}} - b\right) x_{n} x_{n-1}$$

$$\left(\frac{1}{x_{n}} - b\right) - \left(\frac{1}{x_{n-1}} - b\right) = x_{n} + x_{n-1} + x_{n$$

Math 4329, Test I (\mathcal{Q})

Name ______

(L(x) = -1 x2

1. a. If f(x) = ln(cos(x)), find the Taylor polynomial $T_2(x)$ of degree 2 which matches f, f' and f'' at a = 0.

$$f(x) = h(cor(x)) f(x) = 0$$

$$f'(x) = -\frac{\sin x}{\cos x} f'(x) = 0$$

$$f''(x) = -\frac{1}{\cos^2 x} f''(x) = 1$$

b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_2(x)|$ in the interval -0.01 < x < 0.01.

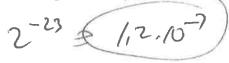
$$2 \int_{-\infty}^{\infty} |x| = -\frac{2\sin x}{\cos^3 x}$$

$$emor \leq \left| \frac{-2\sin x}{\cos^3 x} \right| \leq \frac{2\sin^3 x}{(\cos^3 x)^3} \left| \frac{(\cos^3 x)^3}{(\cos^3 x)^3} \right| \leq \frac{3.3\sqrt{6}}{(\cos^3 x)^3}$$

2. IEEE single precision floating point numbers are stored in a 32-bit word, which includes 1 sign bit, 8 bits for the exponent, and 23 bits for the mantissa (significand). Assuming a normalized binary form is used $(1.xxxxx..._2 * 2^e)$ approximately what are:

a. the overflow limit (largest positive number)

b. the machine precision (smallest $\epsilon > 0$ such that $1 + \epsilon > 1$)





a. For what values of a will the iteration $x_{n+1} = x_n + a * sin(x_n)$ 3. converge for x_0 sufficiently close to the root $r = \pi$?

b. For what value of a will this iteration converge at least quadratically?



4. Estimate the experimental order of convergence for a root finder with errors in 3 consecutive iterations of 0.05, 0.001 and 0.000 000 7.

$$00000007 = M(0.001)^{\alpha}$$
 $0.001 = M(0.001)^{\alpha}$

5. $r = \frac{1}{a}$ is a root of $f(x) = \frac{1}{x} - a$. Write Newton's iteration for finding this root, in a form where no divisions are required; thus this formula can be used to find $\frac{b}{a} = b(\frac{1}{a})$ on a computer which cannot do divisions.

$$X_{n+1} = X_n - \frac{1}{X_n - \alpha} = X_n + (\frac{1}{X_n} - \alpha) X_n^2$$

$$\frac{1}{X_n} = \frac{1}{X_n} = \frac{1}{X_n} - \alpha X_n^2$$

6. For the secant method, $e_{n+1} \approx Me_n e_{n-1}$. If the order of the secant method is α (ie, $e_{n+1} \approx Ce_n^{\alpha}$, and thus also $e_n \approx Ce_{n-1}^{\alpha}$), find α from this.

$$x = 1 + \frac{1}{x}$$

$$\alpha = \frac{1 \pm 51 + 8}{2} = \frac{1 \pm 55}{2}$$

Math 4329, Test I

a. If $f(x) = x^3 + 2x$, find the Taylor polynomial $T_2(x)$ of degree 2 which matches f, f' and f'' at a=2.

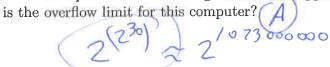
f(2) = 12 $f(x) = x^3 + 2x$ f'(2) = 14 $f'(3) = 3x^2 + 2x$ F11 = 6x

T26) = f(2)+f(h)(x2)+f"(2)(x-2) =12 + 14(x-2) + 6 (x-2)

b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_2(x)|$ in the interval $0 \le x \le 4$.

 $\leq \left| \frac{6}{6} (x-2)^{3} \right| \leq 2^{3} \in 8$

- 2. Computer A stores floating point numbers in a 128-bit word, which includes 1 sign bit, 31 bits for the exponent, and 96 bits for the mantissa. Computer B stores floating point numbers in a 128-bit word, with 1 sign bit, 13 bits for the exponent, and 114 bits for the mantissa.
 - a. Which computer can handle larger numbers? Approximately what



b. Which computer has higher accuracy? Approximately how many significant decimal digits of accuracy does this computer have?





3. If the fixed point iteration $x_{n+1} = x_n + cf(x_n)$ is used near a root r of f(x) = 0, how should the constant c be chosen in order to ensure the fastest convergence?

st convergence?
$$g' = 1 + cf(x)$$

$$g'(r) = 1 + cf'(r) = 0$$

$$c = -1$$

$$f'(r)$$

4. If Newton's method is used to find a root of $f(x) \equiv (x-3)^4 = 0$, for what values of x_0 is convergence to the root r = 3 guaranteed? Hint:

Newton's method can be written in the form $x_{n+1} = g(x_n)$, what is g(x)?

or:

$$g(x)$$
?
 $X_{A+1} = X_1 - \frac{(X_1 - 3)^4}{4(X_1 - 3)^3} = X_1 - \frac{1}{4(X_1 - 3)^3}$
 $X_{A+1} = X_1 - \frac{1}{4(X_1 - 3)^3} = X_1 - \frac{1}{4(X_1 - 3)^3}$
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5. If a = -1000, b = 1000 and f(a) and f(b) have opposite signs, how many bisection iterations are required to find a root between a and b to an accuracy of 10^{-12} ?

$$\frac{2000}{2^{1}} = 10^{-12}$$

$$1 = \ln \left(\frac{2000}{10^{-12}}\right)$$

$$1 = \ln \left(\frac{2000}{10^{-12}}\right)$$

6. A root-finder produces approximations $x_3 = 6.01, x_4 = 6.0001, x_5 = 6.000\ 000\ 06$ when applied to $f(x) \equiv x^2 - 36 = 0$. Estimate the experimental order of convergence. What method have we studied that has approximately this order?

- 7. Consider that fixed-point iteration $x_{n+1} = 2x_n(1-x_n)$.
 - a. What are the two roots (points r such that if $x_n = r$, x_{n+1} will still equal r)? $r = 2r(f-r) \qquad r = 6$
 - b. Analyze each root to determine if the iteration will converge (and if so, with what order) when you start close to that root.

$$g(x) = 2x - 2x^{2}$$

$$g'(x) = 2 - 4x$$

$$g'(0) = 2 \qquad \text{diverger near } r = 0$$

$$g'(t) = 0 \qquad \text{conveyer quadratically near } r = \frac{1}{2}$$

$$g'' = -4$$

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a. If $f(x) = x^4 - 10x^3 + x + 3$, find the Taylor polynomial $T_2(x)$ of degree 2 which matches f, f' and f'' at a = 1.

$$f(x) = x^{4} - (0x^{3} + x + 3) \qquad f(1) = -5$$

$$f(x) = 4x^{3} - 30x^{2} + 1 \qquad f'(1) = -25$$

$$f'(x) = 12x^{2} - 60x \qquad f''(1) = -48$$

$$f'''(x) = 24x - 60$$

b. Use the Taylor remainder formula to get a reasonable bound on the maximum of the error $|f(x) - T_2(x)|$ for 0.5 < x < 1.5.

$$\left|\frac{f'''(y)}{31}(x)|^{3}\right| = \left|\frac{2x - 60}{6}(.5)^{3}\right| = \frac{48}{6}(\frac{1}{8}) = 1$$

May at $c = 0.5$

2. IEEE double precision floating point numbers are stored with one sign bit (0 for positive numbers), 11 exponent bits (1023 + e is stored in binary form here, where e is the exponent of 2), and then 52 bits for the mantissa. Show exactly how 131.125₁₀ would be stored (in either binary or hexadecimal form).

131,125 = 10000011,001, 1.0000011001.200 0/1000000110/00001100100... = 10000000110



3. If
$$f(x) = (x-5)^m$$
 (m is a positive integer) and Newton's method is used to find the root r=5, this can be thought of as a fixed point iteration $x_{n+1} = g(x_n)$. Calculate $g'(5)$ and tell what values of m will cause Newton's method to converge linearly, and what values will cause it to converge even faster. For what range of starting values x_0 will Newton's method converge, if m=3?

$$x_{nel} = x_n - \frac{(x_n - s)^n}{m(x_n - s)^{n-1}} = x_n - \frac{1}{m}(x_n - s)$$

$$g'(x) = 1 - \frac{1}{m} \quad g'(s) = 1 - \frac{1}{m} \quad m = 1 \quad fait$$

$$m > 1 \quad linearly$$

$$all x_0$$

4. Estimate the experimental order of convergence for a root finder with errors in 3 consecutive iterations of 10^{-2} , 10^{-5} and 10^{-10} .

$$10^{-5} = 0(0^{-2})^{10} = 10^{5} = 10^{30} = 10^{30}$$



5. Show how Newton's method could be used to find $b^{\frac{1}{5}}$ without doing anything other than add, subtract, multiply and divide.

$$f(x) = x^{5} - b = 0$$

 $x_{nq} = x - \frac{x^{5} - b}{5x^{4}} = \frac{4}{5x^{4}} + \frac{b}{5x^{4}}$

3

6. Write out the secant iteration for finding a root of $f(x) = \frac{1}{x} - b$, where no divisions are done, thus it could be used to find the root $\frac{1}{b}$ on a computer that can't divide.

no divisions are done, thus it could be used to computer that can't divide.

$$\begin{array}{c}
X_{n+1} = X_n - X_n \\
X_n - X_n - X_n
\end{array}$$

$$\begin{array}{c}
X_n - X_n \\
X_n - X_n
\end{array}$$