Math 4329, Test I (e)

Name Key

1. a. If $f(x) = ln(cos(x))$, find the Taylor polynomial $T_2(x)$ of degree 2 which matches f, f' and f'' at $a = 0$.

$f(x) = \ln(\cos x)$ $\quad f(0) = 0$

$f'(x) = -\frac{\sin x}{\cos x}$ $\quad f'(0) = 0$

$f''(x) = \frac{-1}{\cos^2 x}$ $\quad f''(0) = -1$

$T_2(x) = \frac{-x^2}{2}$

$f'''(x) = \frac{-2\sin x}{\cos^3 x}$

b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_2(x)|$ at $x = 0.1$.

$$|f(x) - T_2(x)| = \left|\frac{f'''(\xi)}{3!}x^3\right| = \left|\frac{1}{6}\frac{2\sin \xi}{\cos^3 \xi}(0.1)^3\right| \le \frac{2(\sin 0.1)}{6\cos^3(0.1)}(0.1)^3 = 3.38 \cdot 10^{-5}$$

2. A certain computer stores floating point numbers in a 128-bit word. If a floating point number is written in normalized binary form ($1.xxxxx..._2 * 2^e$), it is stored using one sign bit (0 if the number is positive), then $e + 4095$ is stored in binary in the next 13 bits, and then the mantissa xxxxx... is stored in the final 114 bits. Show exactly how the number -12.25 would be stored on this computer. Also: approximately how many **decimal** digits of accuracy does this machine have?

$-12.25 = 1100.01_2 = 1.10001 \cdot 2^3 \qquad e = 3$

| 1 | 1 00 0000 0000 10 | 1 0001 00 ... |

4098

hex = C00A20...0

3. Compute the experimental order of convergence for a root finder with errors in 3 consecutive iterations of $10^{-5}, 10^{-7}$ and 10^{-14}.

$$\alpha = \frac{\ln\left(\frac{10^{-7}}{10^{-14}}\right)}{\ln\left(\frac{10^{-5}}{10^{-7}}\right)} = \frac{\ln 10^7}{\ln 10^2} = \frac{7}{2} = 3.5$$

4. The fixed point iteration $x_{n+1} = x_n + sin(x_n)$ has roots at $r = n\pi$ for any integer n. Will this iteration converge if you start very close to the root $r = 0$? Will it converge if you start near the root $r = \pi$? In both cases, if it does converge, what is the order of convergence?

	$r=0$	$r=\pi$
$g(x) = x + \sin x$		
$g'(x) = 1 + \cos x$	2	0
$g''(x) = -\sin x$	—	0
$g'''(x) = -\cos x$	—	1

$r=0$ Diverges

$r=\pi$ converges

order $=3$

5. Show how Newton's method could be used to find $b^{\frac{1}{m}}$ for $b > 0$, where m is a positive integer, without doing anything other than add, subtract, multiply and divide.

$$f(x) = x^m - b = 0$$

$$x_{n+1} = x_n - \frac{x_n^m - b}{m x_n^{m-1}} = \left(1 - \frac{1}{m}\right)x_n + \frac{b}{m x_n^{m-1}}$$

6. Write the secant iteration for solving $f(x) = 1/x - b = 0$, in a form where no divisions are required (thus this iteration could be used to compute $1/b$ on a computer which cannot do divisions).

$$x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - b\right)}{\frac{\left(\frac{1}{x_n} - b\right) - \left(\frac{1}{x_{n-1}} - b\right)}{(x_n - x_{n-1})}}$$

$$x_{n+1} = x_n + x_{n-1} - b x_n x_{n-1}$$

Math 4329, Test I (f)

Name Key

1. a. If $f(x) = x^5 + 2x^2$, find the Taylor polynomial $T_3(x)$ of degree 3 which matches f, f', f'' and f''' at $a = 1$.

$f(1) = 3$ $\quad f(x) = x^5 + 2x^2$

$f'(1) = 9$ $\quad f' = 5x^4 + 4x$

$f''(1) = 24$ $\quad f'' = 20x^3 + 4$

$f'''(1) = 60$ $\quad f''' = 60x^2$

$f^{iv} = 120x$

3

$$T_3(x) = 3 + 9(x-1) + \frac{24}{2}(x-1)^2 + \frac{60}{3!}(x-1)^3$$

b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_3(x)|$ in the interval $-0.5 \le x \le 1.5$.

3

$$|f(x) - T_3(x)| = \left|\frac{f^{iv}(\xi)}{4!}(x-1)^4\right| = \left|\frac{120\,\xi}{4!}(x-1)^4\right| \le \frac{120(1.5)}{24}(1.5)^4 = 37.97$$

(or 25.31)

2. Write the quadratic formula root $[-b+\sqrt{b^2-4ac}]/(2a)$ in a form so that there are no serious problems with roundoff error, when b is positive and very large compared to ac.

2

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \cdot \frac{-b-\sqrt{b^2-4ac}}{-b-\sqrt{b^2-4ac}} = \frac{4ac}{(2a)(-b-\sqrt{b^2-4ac})}$$

$$= \frac{2c}{-b-\sqrt{b^2-4ac}}$$

3. Write out (and simplify) a *secant* method iteration, used to find $\sqrt{b}$, which does only basic arithmetic (add, subtract, mutliply and divide, no square roots).

$f(x) = x^2 - b = 0$

3

$$x_{n+1} = x_n - \frac{x_n^2 - b}{\dfrac{(x_n^2-b)-(x_{n-1}^2-b)}{(x_n - x_{n-1})}} = x_n - \left(\frac{x_n^2 - b}{x_n + x_{n-1}}\right)$$

4. If Newton's method is used to find a root of $f(x) \equiv (x-3)^7 = 0$,

   a. Will Newton's method converge for x_0 close to the root $r = 3$? Explain.

   $e_{n+1} = \frac{6}{7} e_n$

   $$x_{n+1} = x_n - \frac{(x_n-3)^7}{7(x_n-3)^6} = x_n - \frac{1}{7}(x_n-3)$$

   1

   b. What is the order of convergence, if it converges?

   $\alpha = 1$

   1

   c. Will Newton's method converge for all x_0? Explain.

   $$e_n = \left(\frac{6}{7}\right)^n e_0 \to 0 \quad \text{any } x_0$$

   1

5. A certain computer stores floating point numbers in a 32-bit word, which includes 1 sign bit, 9 bits for the exponent, and 22 bits for the mantissa. *Approximately*

   a. What is the underflow limit (smallest positive number)?

   2^{-256}

   1

   b. What is the overflow limit (largest positive number)?

   2^{256}

   1

   c. What is the machine precision (smallest positive number such that $1 + \epsilon > 1$)?

   2^{-22}

   1

6. If the fixed point iteration $x_{n+1} = x_n + cf(x_n)$ is used near a root r of $f(x) = 0$, how should the constant c be chosen in order to ensure the fastest convergence?

   $g(x) = x + cf(x)$

   $g'(x) = 1 + cf'(x)$

   $g'(r) = 1 + cf'(r) = 0$

   $c = -\frac{1}{f'(r)}$

   4

Math 4329, Test I (g)

Name Key

1. a. If $f(x) = 2x^4 + x^3$, find the Taylor polynomial $T_3(x)$ of degree 3 which matches f, f', f'' and f''' at $a = 1$.

3

$$T_3(x) = 3 + 11(x-1) + \frac{30}{2}(x-1)^2 + \frac{54}{6}(x-1)^3$$

$f(x) = 2x^4 + x^3$
$f' = 8x^3 + 3x^2$
$f'' = 24x^2 + 6x$
$f''' = 48x + 6$

$f^{IV} = 48$

b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_3(x)|$ in the interval $-1 \le x \le 3$.

2

$$|f(x) - T_3(x)| = \left|\frac{f^{IV}(\xi)}{4!}(x-1)^4\right| = \left|\frac{48}{24}(x-1)^4\right| \le 2 \cdot 2^4 = 32$$

2. Write the quadratic formula root $[-b + \sqrt{b^2 - 4ac}]/(2a)$ in a form so that there are no serious problems with roundoff error, when b is positive and very large compared to ac.

3

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}\left(\frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}\right) = \frac{2c}{-b - \sqrt{b^2 - 4ac}}$$

3. If Newton's method is used to find a root of $f(x) \equiv (x-3)^5 = 0$, for what range of starting values x_0 will we get convergence to the root $r = 3$? What is the order of convergence of Newton's method in this case?

3

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n - 3)^5}{5(x_n-3)^4} = x_n - \frac{1}{5}(x_n - 3)$$

$$e_{n+1} = e_n\left(1 - \frac{1}{5}\right) = \frac{4}{5}e_n$$

Converges linearly for all x_0

4. If $a = -10, b = 10$ and $f(a)$ and $f(b)$ have opposite signs, about how many bisection iterations are required to find a root between a and b to an accuracy of 10^{-10}?

3

$$\frac{b-a}{2^N} = 10^{-10} \qquad 2^N = \frac{20}{10^{-10}} = 2 \cdot 10^{11}$$

$$N \approx \frac{\ln(2 \cdot 10^{11})}{\ln 2} = 38$$

5. Compute the experimental order of convergence for a root finder with errors in 3 consecutive iterations of $10^{-4}, 10^{-7}$ and 10^{-14}.

3

$$10^{-7} = \mu(10^{-4})^{\alpha}$$

$$10^{-14} = \mu(10^{-7})^{\alpha} \qquad \alpha = 2\tfrac{1}{3}$$

$$10^{7} = (10^{3})^{\alpha} \qquad 3\alpha = 7$$

6. The fixed point iteration $x_{n+1} = x_n + sin(x_n)$ has roots at $r = n\pi$ for any integer n. For which of these roots (which values of n) will the interation converge? What will be the order of convergence for these roots?

3

$$g(x_n) = x_n + \sin(x_n)$$

$$g'(x) = 1 + \cos(x) \qquad n \text{ odd}: g'(n\pi) = 0 \text{ conv}$$

$$g''(x) = -\sin(x) \qquad n \text{ even}: g'(n\pi) = 2 \text{ diverge}$$

$$g'''(x) = -\cos(x)$$

n odd $g'(n\pi) = 0$

$g''(n\pi) = 0$

$g'''(n\pi) \neq 0$ so

cubic

Math 4329, Test I (h)

Name Key

1. a. If $f(x) = e^{2x}$, find the Taylor polynomial $T_n(x)$ of degree n which matches $f, f', f''...f^{(n)}$ at $a = 0$.

$f^{(n)}(x) = 2^n e^{2x}$

$f^n(0) = 2^n$

2

$$T_n(x) = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3!} + \dots \frac{2^n x^n}{n!}$$

b. Use the Taylor remainder formula to get a reasonable bound (in terms of n) on the error $|f(x) - T_n(x)|$ at $x = 3$.

2

$$|f(x) - T_n(x)| = \left|\frac{f^{n+1}(c)\, x^{n+1}}{(n+1)!}\right| = \frac{2^{n+1} e^{2\xi}\, 3^{n+1}}{(n+1)!} \le \frac{e^6\, 6^{n+1}}{(n+1)!}$$

2. Computer A stores floating point numbers in a 96-bit word, which includes 1 sign bit, 11 bits for the exponent, and 84 bits for the mantissa. Computer B stores floating point numbers in a 96-bit word, with 1 sign bit, 25 bits for the exponent, and 70 bits for the mantissa.

1

a. Which computer can handle larger numbers? B

b. Which computer has higher accuracy? Approximately how many significant **decimal** digits of accuracy does this computer have?

2

A $\simeq 2^{84} = 10^{25}$ 25 digits

3 3. A root-finder produces approximations $x_3 = 5.01, x_4 = 5.0001, x_5 = 5.000\,000\,06$ where one root is $r = 5$? Estimate the experimental order of convergence. What method have we studied that has approximately this order?

$.0001 = M(.01)^\alpha$

$.000\,000\,06 = M(.0001)^\alpha$

$1666 = 100^\alpha$

$\alpha = 1.61$ secant

3 4. Write $\frac{\sqrt{4+x}-2}{x}$ in a form where there is no serious problem with roundoff, when $x \approx 0$.

$$\frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = \frac{(4+x)-4}{x(\sqrt{4+x}+2)} = \frac{1}{\sqrt{4+x}+2}$$

2 5. a. Newton's method is sometimes used to find $\frac{1}{b}$ by computing the root of $f(x) = b - \frac{1}{x}$. Write the Newton iteration in a form where no divisions are required (thus we can find $\frac{1}{b}$ without doing any divisions).

$$x_{n+1} = x_n - \frac{b - \frac{1}{x_n}}{\left(\frac{1}{x_n^2}\right)} = x_n(2 - bx_n)$$

2 b. Same as (5a) but use the secant method.

$$x_{n+1} = x_n - \frac{\left(b - \frac{1}{x_n}\right)}{\left[\frac{\left(b-\frac{1}{x_n}\right)-\left(b-\frac{1}{x_{n-1}}\right)}{x_n - x_{n-1}}\right]} = x_n + x_{n-1} - bx_n x_{n-1}$$

3 6. The polynomial $x^3 - x^2 - x - 1$ has one real root, at $x = 1.839$. We can write $x^3 - x^2 - x - 1 = 0$ in the form $x^3 = x^2 + x + 1$, or $x = 1 + \frac{1}{x} + \frac{1}{x^2}$ and try the iteration $x_{n+1} = 1 + \frac{1}{x_n} + \frac{1}{x_n^2}$.

Will this converge, for x_0 near 1.839? Justify your answer without actually doing any iterations.

$g(x) = 1 + \frac{1}{x} + \frac{1}{x^2}$

$g'(x) = -\frac{1}{x^2} - \frac{2}{x^3}$

yes

$g'(1.839) = -\frac{1}{(1.839)^2} - \frac{2}{(1.839)^3} = -0.617$

Math 4329, Test I (i)

Name Key

1. a. If $f(x) = ln(1+sin(x))$, find the Taylor polynomial $T_2(x)$ of degree 2 which matches f, f' and f'' at $a = 0$.

$f(0)=0 \quad f(x) = \ln(1+\sin x)$

$f'(0)=1 \quad f' = \frac{1}{1+\sin x}(\cos x)$

$f''(0)=-1 \quad f'' = \frac{-1}{1+\sin x}$

$f''' = \frac{\cos x}{(1+\sin x)^2}$

$T_2(x) = x - \frac{x^2}{2}$

3

b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_2(x)|$ at $x = 0.1$.

2 $|f(x) - T_2(x)| = \left|\frac{f'''(c)}{3!}x^3\right| = \left|\frac{\cos(c)}{(1+\sin c)^2}\frac{(0.1)^3}{6}\right| \le \frac{.001}{6} = 1.66 \cdot 10$

$0 \le c \le .1$

2. A certain computer stores floating point numbers in a 96-bit word. If a floating point number is written in normalized binary form ($1.xxxxx..._2 * 2^e$), it is stored using one sign bit (0 if the number is positive), then $e + 2047$ is stored in binary in the next 12 bits, and then the mantissa xxxxx... is stored in the final 83 bits. Show exactly how the number -17.125 would be stored on this computer. Also, approximately how many **decimal** digits of accuracy does this machine have?

4

$-17.125 = -10001.001_2 = -1.0001001 \cdot 2^4$

exp-12 mantissa-83

| 1 | 100000000011 | 0001001 0000... 0000 |

$2047+4$
$=2051$

$\epsilon_{mach} \approx 2^{-83} \cong 10^{-25}$

25 digits

C0189 0... 0

3. The polynomial $x^2 - x - 1$ has two roots, $r_1 = -0.618, r_2 = 1.618$. We can write $x^2 - x - 1 = 0$ in the form $x^2 = x + 1$, and then $x = 1 + \frac{1}{x}$ and try the iteration $x_{n+1} = 1 + \frac{1}{x_n}$.

Will this converge, for x_0 near $r_1 = -0.618$? Justify your answer **theoretically**. Will it converge near $r_2 = 1.618$?

4

$g(x) = 1 + \frac{1}{x}$ $\quad g'(-0.618) = -2.62$ no, diverges

$g'(x) = \frac{-1}{x^2}$ $\quad g'(1.618) = 0.382$ yes, converges

4. Estimate the experimental order of convergence for a root finder with errors in 3 consecutive iterations of $3 * 10^{-3}, 2 * 10^{-4}$ and $5 * 10^{-8}$.

3

$2 \cdot 10^{-4} = M(3 \cdot 10^{-3})^{\alpha}$ $\quad 4000 = 15^{\alpha}$

$5 \cdot 10^{-8} = M(2 \cdot 10^{-4})^{\alpha}$ $\quad \alpha = \frac{\ln 4000}{\ln 15} = 3.06$

5. a. $r = \sqrt{a}$ is a root of $f(x) = x^2 - a = 0$. Write Newton's iteration for finding this root.

2

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

b. Given that, for Newton's method:

$$x_{n+1} - r = \frac{f''(c_n)}{2f'(x_n)}(x_n - r)^2$$

where c_n is between x_n and the root r, show that the iteration in 5a will converge provided $x_0 > \sqrt{a}/3$. (Hint: just show that x_1 will be closer to the root than x_0.)

2

$$e_{n+1} = \left(\frac{2}{2(2x_n)}(x_n - r)\right)e_n$$

should be < 1 in abs. value

$$-1 < \frac{x_n - r}{2x_n} < 1 \Rightarrow 0 < 3x_n - r \leq 4x_n$$

$$\Rightarrow x_n > \frac{r}{3} \quad \text{and } x_n > -r$$

Math 4329, Test I (j)

Name Key

$f' = 4x^3 - 6x^2 + 1$

$f'' = 12x^2 - 12x$

$f''' = 24x - 12$

$f^{iv} = 24$

1. a. If $f(x) = x^4 - 2x^3 + x + 3$, find the Taylor polynomial $T_3(x)$ of degree 3 which matches f, f', f'' and f''' at $a = 1$.

$$T_3(x) = f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2} + f'''(1)\frac{(x-1)^3}{6}$$

$$= 3 + (-1)(x-1) + 2(x-1)^3$$

4

b. Use the Taylor remainder formula to get a reasonable bound (in terms of x) on the error $|f(x) - T_3(x)|$ at x.

$$f(x) - T_3(x) = \frac{f^{iv}(\xi)}{4!}(x-1)^4 = (x-1)^4$$

2. Computer A stores floating point numbers in a 128-bit word, which includes 1 sign bit, 21 bits for the exponent, and 106 bits for the mantissa. Computer B stores floating point numbers in a 128-bit word, with 1 sign bit, 13 bits for the exponent, and 114 bits for the mantissa.

4

a. Which computer can handle larger numbers? Approximately what is the overflow limit for this computer? A

$\approx 2^{(2^{20})} \approx 10^{315{,}000}$

b. Which computer has higher accuracy? Approximately how many significant **decimal** digits of accuracy does this computer have?

B $\quad 2^{-114} = 5 \cdot 10^{-35}$

35

3. Compute the experimental order of convergence for a root finder with errors in 3 consecutive iterations of 0.03, 0.002 and 0.00001.

$$\frac{.002 = M(.03)^\alpha}{.00001 = M(.002)^\alpha} \qquad 200 = 15^\alpha \quad \alpha = 1.96$$

3

4. Will the iteration $x_{n+1} = 4x_n(1 - x_n)$ converge when x_0 is sufficiently close to the root $r = \frac{3}{4}$? (Justify your answer theoretically, without actually iterating the formula.) If it converges, what is the order?

$$g(x) = 4x - 4x^2$$
$$g'(x) = 4 - 8x \qquad \text{no}$$
$$g'(\tfrac{3}{4}) = -2$$

3

5. Show how Newton's method could be used to find $b^{p/q}$, where p, q are integers, and $b > 0$, without doing anything other than add, subtract, multiply and divide.

$$f(x) = x^q - b^p = 0$$

$$x_{n+1} = x_n - \frac{x_n^q - b^p}{q x_n^{q-1}} = \left(1 - \frac{1}{q}\right)x_n + \frac{b^p}{q x_n^{q-1}}$$

3

6. For the secant method, $e_{n+1} \approx M e_n e_{n-1}$. If the order of the secant method is α (ie, $e_{n+1} \approx C e_n^\alpha$), show that α must satisfy $\alpha = 1 + 1/\alpha$.

$$e_n = C e_{n-1}^\alpha \quad \text{so} \quad e_{n+1} = M e_n \left(\frac{e_n}{C}\right)^{\frac{1}{\alpha}} = \frac{M}{C^{\frac{1}{\alpha}}} e_n^{(1+\frac{1}{\alpha})}$$

$$\left(\frac{e_n}{C}\right)^{\frac{1}{\alpha}} = e_{n-1}$$

$$\| \qquad C e_n^{\alpha}$$

3

$$\alpha = 1 + \frac{1}{\alpha}$$

Math 4329, Test I (K)

Name ____Key____

1. a. If $f(x) = e^{x/2}$ use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_n(x)|$ for $-2 \le x \le 2$, where $T_n(x)$ is the Taylor polynomial of degree n for $f(x)$, at $a = 0$.

3

$$\left| \frac{f^{n+1}(c)}{(n+1)!} x^{n+1} \right| \le \frac{\left(\frac{1}{2}\right)^{n+1} e^{c/2}}{(n+1)!} 2^{n+1} \le \frac{e^1}{(n+1)!}$$

b. Approximately how large does n need to be so that this error bound is less than 10^{-10}?

1

$$\frac{e}{(n+1)!} = 10^{-10} \qquad (n+1)! = 2.7 \cdot 10^{10} \Rightarrow n \approx 13$$

2. A certain computer stores floating point numbers in a 128-bit word, which includes 1 sign bit, 17 bits for the exponent, and 110 bits for the mantissa (significand). Assuming a normalized binary form is used $(1.xxxxx..._2 * 2^e)$ **approximately** what are:

a. the overflow limit (largest positive number)

2

$$2^{(2^{16})} = 2^{65536} \approx 10^{19728}$$

b. the machine precision (smallest $\epsilon > 0$ such that $1 + \epsilon > 1$)

2

$$2^{-110} \approx 7.7 \cdot 10^{-34}$$

3. Consider that fixed-point iteration $x_{n+1} = 2.5x_n(1 - x_n)$.

   a. What are the two roots (points r such that if $x_n = r$, x_{n+1} will still equal r)?

   $\cancel{x} = 2.5x(1-x)$ $\quad$ $x=0$ or $x=0.6$

   2 $\quad$ $0.4 = 1-x$

   $x = 0.6$

   b. Analyze each root to determine if the iteration will converge (and if so, with what order) when you start close to that root.

   3 $\quad$ $g(x) = 2.5x - 2.5x^2$ $\quad$ $g'(0) = 2.5$ diverges $r=0$

   $g'(x) = 2.5 - 5x$ $\quad$ $g'(0.6) = -0.5$ converges $r=0.6$ linearly

4. Estimate the experimental order of convergence for a root finder with errors in 3 consecutive iterations of 10^{-5}, 10^{-7} and 10^{-14}.

   3 $\quad$ $10^{-7} = M(10^{-5})^{\alpha}$ $\quad$ $10^{7} = 10^{2\alpha}$ $\quad$ $\alpha = 3.5$

   $10^{-14} = M(10^{-7})^{\alpha}$

5. The root of $f(x) \equiv \frac{1}{x} - b = 0$ is $x = \frac{1}{b}$.

   a. Write Newton's iteration for solving $f(x) = 0$ in a form so that no divisions are required; thus providing a way to find $\frac{1}{b}$ without doing any divisions.

   2 $\quad$ $x_{n+1} = x_n - \dfrac{\frac{1}{x_n} - b}{\left(-\frac{1}{x_n^2}\right)} = x_n + \left(\frac{1}{x_n} - b\right)x_n^2 = x_n(2 - bx_n)$

   b. Same problem, but use the secant iteration.

   2 $\quad$ $x_{n+1} = x_n - \dfrac{\left(\frac{1}{x_n} - b\right)}{\left(\dfrac{\left(\frac{1}{x_n} - b\right) - \left(\frac{1}{x_{n-1}} - b\right)}{x_n - x_{n-1}}\right)} = x_n + \left(\frac{1}{x_n} - b\right)x_n x_{n-1}$

   $x_{n+1} = x_n + x_{n-1} - b x_n x_{n-1}$

Math 4329, Test I (2)

Name Key

1. a. If $f(x) = ln(cos(x))$, find the Taylor polynomial $T_2(x)$ of degree 2 which matches f, f' and f'' at $a = 0$.

$f(x) = \ln(\cos(x))$ $\quad f(0) = 0$

$f'(x) = \frac{-\sin x}{\cos x}$ $\quad f'(0) = 0$

$f''(x) = \frac{-1}{\cos^2 x}$ $\quad f''(0) = -1$

2

$T_2(x) = -\frac{1}{2}x^2$

b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_2(x)|$ in the interval $-0.01 < x < 0.01$.

2 $f'''(x) = -\frac{2\sin x}{\cos^3 x}$

error $\le \left| \frac{-2\sin\psi}{\cos^3\psi} \frac{x^3}{6} \right| \le \frac{2\sin 0.01}{(\cos 0.01)^3} \frac{(.01)^3}{6} = 3.3 \cdot 10^{-9}$

2. IEEE single precision floating point numbers are stored in a 32-bit word, which includes 1 sign bit, 8 bits for the exponent, and 23 bits for the mantissa (significand). Assuming a normalized binary form is used ($1.xxxxx..._2 * 2^e$) **approximately** what are:

a. the overflow limit (largest positive number)

2

$2^{128} = 3 \cdot 10^{38}$

b. the machine precision (smallest $\epsilon > 0$ such that $1 + \epsilon > 1$)

$2^{-23} = 1.2 \cdot 10^{-7}$

2

3. a. For what values of a will the iteration $x_{n+1} = x_n + a * sin(x_n)$ converge for x_0 sufficiently close to the root $r = \pi$?

$g'(x) = 1 + a\cos(x)$ $\quad g'(\pi) = 1-a \quad -1 < 1-a < 1$

$0 < a < 2$

2

b. For what value of a will this iteration converge at least quadratically?

$a = 1$

1

4. Estimate the experimental order of convergence for a root finder with errors in 3 consecutive iterations of 0.05, 0.001 and 0.000 000 7.

$.000\,000\,7 = M(0.001)^\alpha$

$0.001 = M(.05)^\alpha$

$1428 = 50^\alpha$

$\alpha = 1.85$

3

5. $r = \frac{1}{a}$ is a root of $f(x) = \frac{1}{x} - a$. Write Newton's iteration for finding this root, in a form where no divisions are required; thus this formula can be used to find $\frac{b}{a} = b(\frac{1}{a})$ on a computer which cannot do divisions.

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} = x_n + \left(\frac{1}{x_n} - a\right)x_n^2$$

$$x_{n+1} = 2x_n - ax_n^2$$

3

6. For the secant method, $e_{n+1} \approx Me_ne_{n-1}$. If the order of the secant method is α (ie, $e_{n+1} \approx Ce_n^\alpha$, and thus also $e_n \approx Ce_{n-1}^\alpha$), find α from this.

$$Ce_n^\alpha = Me_n\left(\frac{e_n}{C}\right)^{\frac{1}{\alpha}} = \frac{M}{C^{\frac{1}{\alpha}}} e_n^{1+\frac{1}{\alpha}}$$

3

$\alpha = 1 + \frac{1}{\alpha}$

$\alpha^2 - \alpha - 1 = 0$

$$\alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$= 1.618$

(m)

Math 4329, Test I

Name Key

1. a. If $f(x) = x^3 + 2x$, find the Taylor polynomial $T_2(x)$ of degree 2 which matches f, f' and f'' at $a = 2$.

$f(2) = 12$
$f'(2) = 14$
$f''(2) = 12$

$f(x) = x^3 + 2x$
$f' = 3x^2 + 2$
$f'' = 6x$

$T_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)(x-2)^2}{2}$
$= 12 + 14(x-2) + 6(x-2)^2$

b. Use the Taylor remainder formula to get a reasonable bound on the error $|f(x) - T_2(x)|$ in the interval $0 \le x \le 4$.

$f''' = 6$

$\left|\frac{f'''(c)}{3!}(x-2)^3\right| \le \left|\frac{6}{6}(x-2)^3\right| \le 2^3 = 8$

2. Computer A stores floating point numbers in a 128-bit word, which includes 1 sign bit, 31 bits for the exponent, and 96 bits for the mantissa. Computer B stores floating point numbers in a 128-bit word, with 1 sign bit, 13 bits for the exponent, and 114 bits for the mantissa.

a. Which computer can handle larger numbers? Approximately what is the overflow limit for this computer? A

$2^{(2^{30})} \approx 2^{1\,073\,000\,000}$

b. Which computer has higher accuracy? Approximately how many significant **decimal** digits of accuracy does this computer have?

B

$\epsilon \approx 2^{-114} = 5 \cdot 10^{-35}$ ≈ 34 digits

3. If the fixed point iteration $x_{n+1} = x_n + cf(x_n)$ is used near a root r of $f(x) = 0$, how should the constant c be chosen in order to ensure the fastest convergence?

$$g' = 1 + cf'(x_n)$$

$$g'(r) = 1 + cf'(r) = 0$$

$$c = \frac{-1}{f'(r)}$$

4. If Newton's method is used to find a root of $f(x) \equiv (x-3)^4 = 0$, for what values of x_0 is convergence to the root $r = 3$ guaranteed? Hint: Newton's method can be written in the form $x_{n+1} = g(x_n)$, what is $g(x)$?

$$x_{n+1} = x_n - \frac{(x_n-3)^4}{4(x_n-3)^3} = x_n - \frac{1}{4}(x_n - 3)$$

$$x_{n+1} = \frac{3}{4}x_n + \frac{3}{4}$$

$$g'(x_n) = \frac{3}{4} \Rightarrow \text{any } x_0$$

or:

$$x_{n+1} - 3 = x_n - 3 - \frac{1}{4}(x_n - 3)$$

$$e_{n+1} = \frac{3}{4}e_n$$

5. If $a = -1000, b = 1000$ and $f(a)$ and $f(b)$ have opposite signs, how many bisection iterations are required to find a root between a and b to an accuracy of 10^{-12}?

$$\frac{2000}{2^n} = 10^{-12}$$

$$n = \frac{\ln\left(\frac{2000}{10^{-12}}\right)}{\ln 2} \doteq 51$$

6. A root-finder produces approximations $x_3 = 6.01, x_4 = 6.0001, x_5 = 6.000\,000\,06$ when applied to $f(x) \equiv x^2 - 36 = 0$. Estimate the experimental order of convergence. What method have we studied that has approximately this order?

$e_3 = .01$

$e_4 = .0001$

$e_5 = .000\,000\,06$

$e_4 = M e_3^{\alpha}$

$e_5 = M e_4^{\alpha}$

$\frac{e_4}{e_5} = \left(\frac{e_3}{e_4}\right)^{\alpha}$

$1666 = 100^{\alpha}$

$\alpha = 1.61$ secant method

7. Consider that fixed-point iteration $x_{n+1} = 2x_n(1 - x_n)$.

a. What are the two roots (points r such that if $x_n = r$, x_{n+1} will still equal r)?

$r = 2r(1-r)$

$r = 0$

$r = \frac{1}{2}$

b. Analyze each root to determine if the iteration will converge (and if so, with what order) when you start close to that root.

$g(x) = 2x - 2x^2$

$g'(x) = 2 - 4x$

$g'(0) = 2 \rightarrow$ diverges near $r = 0$

$g'(\frac{1}{2}) = 0 \rightarrow$ converges quadratically near $r = \frac{1}{2}$

$g'' = -4$

Math 4329, Test I (0)

Name Key

1. a. If $f(x) = x^4 - 10x^3 + x + 3$, find the Taylor polynomial $T_2(x)$ of degree 2 which matches f, f' and f'' at $a = 1$.

$f(x) = x^4 - 10x^3 + x + 3 \qquad f(1) = -5$

$f'(x) = 4x^3 - 30x^2 + 1 \qquad f'(1) = -25$

$f''(x) = 12x^2 - 60x \qquad f''(1) = -48$

$f'''(x) = 24x - 60$

$T_2(x) = -5 - 25(x-1) - \frac{48}{2}(x-1)^2$

3

b. Use the Taylor remainder formula to get a reasonable bound on the maximum of the error $|f(x) - T_2(x)|$ for $0.5 < x < 1.5$.

$\left|\frac{f'''(\psi)}{3!}(x-1)^3\right| \leq \left|\frac{24\psi - 60}{6}\right|(.5)^3 \leq \frac{48}{6}\left(\frac{1}{8}\right) = 1$

Max at $c = 0.5$

3

2. IEEE double precision floating point numbers are stored with one sign bit (0 for positive numbers), 11 exponent bits ($1023 + e$ is stored in binary form here, where e is the exponent of 2), and then 52 bits for the mantissa. Show exactly how 131.125_{10} would be stored (in either binary or hexadecimal form).

$131.125 = 10000011.001_2 \qquad 1.0000011001 \cdot 2^{7}$ e

$1023 + 7 = 1030$

$= 10000000110$

0 | 10000000110 | 000001100100..

$4 0 6 0 6 4 0 0 0 0 0 0 0 0 0 0_{16}$

3

3. If $f(x) = (x-5)^m$ (m is a positive integer) and Newton's method is used to find the root r=5, this can be thought of as a fixed point iteration $x_{n+1} = g(x_n)$. Calculate $g'(5)$ and tell what values of m will cause Newton's method to converge linearly, and what values will cause it to converge even faster. For what range of starting values x_0 will Newton's method converge, if m=3?

3

$$x_{n+1} = x_n - \frac{(x_n-5)^m}{m(x_n-5)^{m-1}} = x_n - \frac{1}{m}(x_n-5)$$

$$g'(x) = 1 - \frac{1}{m} \quad g'(5) = 1 - \frac{1}{m}$$

m = 1 fast
m > 1 linearly

all x_0

4. Estimate the experimental order of convergence for a root finder with errors in 3 consecutive iterations of $10^{-2}, 10^{-5}$ and 10^{-10}.

2

$$10^{-5} = c(10^{-2})^{\alpha}$$
$$10^{-10} = c(10^{-5})^{\alpha}$$

$$= 10^{5} = 10^{3\alpha} \qquad \alpha = \frac{5}{3}$$

5. Show how Newton's method could be used to find $b^{\frac{1}{5}}$ without doing anything other than add, subtract, multiply and divide.

3

$$f(x) = x^5 - b = 0$$

$$x_{n+1} = x_n - \frac{x_n^5 - b}{5x_n^4} = \frac{4}{5}x_n + \frac{b}{5x_n^4}$$

6. Write out the secant iteration for finding a root of $f(x) = \frac{1}{x} - b$, where no divisions are done, thus it could be used to find the root $\frac{1}{b}$ on a computer that can't divide.

3

$$x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - b\right)}{\frac{\left(\frac{1}{x_n} - b\right) - \left(\frac{1}{x_{n-1}} - b\right)}{(x_n - x_{n-1})}}$$

$$x_{n+1} = x_n + x_{n-1} - b x_n x_{n-1}$$