Math 5329, Test I

Name ___________________________

1. a. Find $T_n(x)$, the Taylor series of degree $n$ for the function $f(x) = \ln(1 + x)$, expanded around $c = 0$.
   (Hint: $f^{(n)}(x) = (-1)^{n-1}(n - 1)!/(1 + x)^n$, for $n \geq 1$.)

b. Find $E_n(x)$, the error in $T_n(x)$, and find a reasonable upper bound on $E_n(1)$.

c. Estimate the number of terms $n$ required for $T_n(x)$ to approximate $f(1) = \ln(2)$ to 5 decimal places accuracy.

d. Would you expect roundoff error to be a serious concern in (c)?
   Why or why not?
   (Hint: $1 + 1/2 + 1/3 + 1/4 + 1/5 + ... + 1/n \approx \ln(n)$, for large $n$.)
2. Estimate the order of convergence of a root-finder that has consecutive errors 0.2, 0.08, 0.00512.

3. If Newton’s method is used to find a root of \( f(x) = x^2 - R \), find bounds on \( x_0 \) for which convergence to the root \( \sqrt{R} \) is guaranteed. (Hint: for Newton’s method, \( e_{n+1} = \frac{1}{2}[f''(\psi_n)/f'(x_n)]e_n^2 \), where \( \psi_n \) is between \( x_n \) and the root.)

4. The golden search method tries to minimize \( f(x) \), where \( f \) is assumed to be unimodal in \( a \leq x \leq b \), by evaluating \( f \) at two points between \( a \) and \( b \), \( p = a + (1 - r) \ast (b - a) \) and \( q = a + r \ast (b - a) \), where \( r = 0.618 \ldots \). If \( f(q) \) is larger than \( f(p) \), the minimum is known to be in the new interval \([a, q]\), otherwise the minimum is known to be in \([p, b]\). Why? Would this algorithm still work if we used \( r = 0.75 \)? What is the advantage of using \( r = 0.618 \ldots \)?
5. Write out the equations used to solve the following system using Newton’s method:

\[ f(x, y) = 1 + x^2 - y^2 + e^x \cos(y) = 0 \]
\[ g(x, y) = 2xy + e^x \sin(y) = 0 \]

6. To solve \( x^2 - 3x - 4 = 0 \) we could write \( x^2 = 3x + 4 \), then \( x = 3 + 4/x \), and iterate with this last formula: \( x_{n+1} = 3 + 4/x_n \). Determine (without actually iterating) if this iteration will converge if we start near the root \( r = 4 \). What if we start near the root \( r = -1 \)?