Math 5329 Final

Name ____________________________

1. If $f(x) = (x-r)^m$, where $m > 1$, show that Newton's method will converge to the multiple root r, no matter where we start, but only linearly.

2. Find the optimum weights w_1, w_2 and sample points r_1, r_2 for the (Gauss) 2 point quadrature rule $\int_0^h f(x)\,dx \approx w_1 h f(r_1 h) + w_2 h f(r_2 h)$. (Hint: You can make some simplifing assumptions first, based on symmetry)

3. a. A root finder gives consecutive errors of $e_6 = 10^{-2}, e_7 = 10^{-5}, e_8 = 10^{-13}$. Estimate the order of the method.

   b. A quadrature method gives an error of 10^{-4} when $h = 10^{-2}$ and 10^{-11} when $h = 10^{-4}$. Estimate the (global) order of the method.

4. a. Is the following method stable? (Justify answer)

$$\frac{U_{k+1}-U_k}{h} = f(t_k, U_k) + f(t_{k-1}, U_{k-1})$$

b. Is the method consistent (with $u' = f(t,u)$)? (Justify answer)

c. Will the finite difference solution converge to the differential equation solution as $h \to 0$?

5. Let $T_3(x)$ be the Taylor polynomial of degree 3 which matches $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$ at $x = 0$, where $f(x) = x^4$. Find the best possible bound on

$$max_{-0.5 \le x \le 0.5} |T_3(x) - f(x)| \le$$

6. Use the inverse power method to determine the smallest eigenvalue of

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

Start the iteration with $(x_0, y_0) = (1, 1)$.

7. Give an example of a linear system $Ax = b$ for which the Jacobi iteration converges, although the matrix A is not diagonal dominant. (Hint: If $A = L + D + U$, the Jacobi method converges if and only if all eigenvalues of $D^{-1}(L + U)$ are less than one in absolute value.)

8. Do one iteration of Newton's method, starting from $(0, 0)$, to solve:

   $f(x, y) = 2x^3 + y - 1 = 0$
   $g(x, y) = -2x + y^3 + 1 = 0$

9. If $p_N(x)$ is the polynomial of degree N which interpolates to $f(x) = e^{\alpha x}$ at $x = 1, 2, 3, ..., N + 1$, use the Lagrange error formula to show that $p_N(0)$ does NOT converge to $f(0)$ as $N \to \infty$, if $\alpha > 1$.

10. Write $[\sqrt{1 + x} - 1]/x$ in a form such that there is not a serious problem with roundoff error when x is close to 0.