Math 5329, Test I (a)

Name Key

1. a. Find $T_n(x)$, the Taylor series of degree n for the function $f(x) = ln(1+x)$, expanded around $c = 0$.
(Hint: $f^{(n)}(x) = (-1)^{n-1}(n-1)!/(1+x)^n$, for $n \geq 1$.)

1 $T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \frac{x^n}{n}$

b. Find $E_n(x)$, the error in $T_n(x)$, and find a reasonable upper bound on $E_n(1)$. $(x > 0)$

2 $|E_n(x)| = \left| \frac{(-1)^n n!}{(1+\epsilon)^{n+1}} \frac{x^{n+1}}{(n+1)!} \right| = \left| \frac{1}{(1+\epsilon)^{n+1}} \frac{x^{n+1}}{n+1} \right| \leq \frac{x^{n+1}}{n+1}$

$|E_n(1)| \leq \frac{1}{n+1}$

c. Estimate the number of terms n required for $T_n(x)$ to approximate $f(1) = ln(2)$ to 5 decimal places accuracy.

1 $n \geq 100{,}000$ $\qquad \frac{1}{n+1} \leq 0.00001$

d. Would you expect roundoff error to be a serious concern in (c)? Why or why not?
(Hint: $1 + 1/2 + 1/3 + 1/4 + 1/5 + \dots + 1/n \approx ln(n)$, for large n.)

1 no. $\ln(2) \doteq \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \frac{1}{99999}\right]$
$- \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots \frac{1}{100000}\right]$

positive term $\leq \ln(100000) \simeq 11$
negative term ≈ 11

2. Estimate the order of convergence of a root-finder that has consecutive errors 0.2, 0.08, 0.00512.

3

$$0.08 = M\,(0.2)^{\alpha} \qquad 15.625 = 2.5^{\alpha}$$
$$0.00512 = M\,(.08)^{\alpha} \qquad \alpha = 3.00$$

3. If Newton's method is used to find a root of $f(x) = x^2 - R$, find bounds on x_0 for which convergence to the root $\sqrt{R}$ is guaranteed.
(Hint: for Newton's method, $e_{n+1} = \frac{1}{2}[f''(\psi_n)/f'(x_n)]e_n^2$, where ψ_n is between x_n and the root.)

3

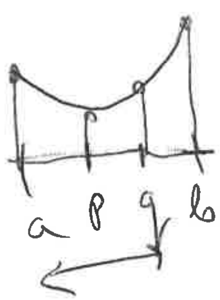
$$e_{n+1} = \frac{1}{2}\,\frac{2}{2x_n}\,e_n^2 = \left[\frac{x_n - \sqrt{R}}{2x_n}\right] e_n$$

$$\left|\frac{x_n - \sqrt{R}}{2x_n}\right| < 1 \quad \text{if} \quad x_0 > \frac{1}{3}\sqrt{R} \qquad \text{or} \quad x_0 > \frac{\sqrt{R}}{\sqrt{3}}$$

$$-1 < \frac{1 - \frac{\sqrt{R}}{x_n}}{2} < 1 \Rightarrow$$

4. The golden search method tries to minimize $f(x)$, where f is assumed to be unimodal in $a \le x \le b$, by evaluating f at two points between a and b, $p = a + (1 - r) * (b - a)$ and $q = a + r * (b - a)$, where $r = 0.618....$ If $f(q)$ is larger than $f(p)$, the minimum is known to be in the new interval $[a, q]$, otherwise the minimum is known to be in $[p, b]$. Why? Would this algorithm still work if we used $r = 0.75$? What is the advantage of using $r = 0.618...$?

3



If $f(q) > f(p)$ the minimum cannot be in (q, b) otherwise f would be increasing for some points to the left of q and decreasing for some points to the right, hence not unimodal. Would still work with $r = 0.75$ but would require two new function evaluation per iteration. With $r^2 = 1 - r$, can use old p as new q in (a, q)

5. Write out the equations used to solve the following system using Newton's method:
$f(x, y) = 1 + x^2 - y^2 + e^x cos(y)$
$g(x, y) = 2xy + e^x sin(y)$

3

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{bmatrix} 2x_n + e^{x_n}(\cos y_n) & -2y_n - e^{x_n}\sin y_n \\ 2y_n + e^{x_n}\sin y_n & 2x_n + e^{x_n}\cos y_n \end{bmatrix}^{-1} \begin{bmatrix} 1 + x_n^2 - y_n^2 + e^{x_n}\cos y_n \\ 2x_n y_n + e^{x_n}\sin y_n \end{bmatrix}$$

6. To solve $x^2 - 3x - 4 = 0$ we could write $x^2 = 3x + 4$, then $x = 3 + 4/x$, and iterate with this last formula: $x_{n+1} = 3 + 4/x_n$. Determine (without actually iterating) if this iteration will converge if we start near the root $r = 4$. What if we start near the root $r = -1$?

3

$$x_{n+1} = 3 + \frac{4}{x_n} \equiv g(x_n)$$

$$g'(x) = -\frac{4}{x^2} \quad g'(4) = -0.25 \quad \text{converges}$$

$$g'(-1) = -4 \quad \text{diverges}$$

Math 5329, Test I (b)

Name ____Key____

1. a. Write the Taylor polynomial $T_n(x)$ of degree n for the function $f(x) = cos(x)$, expanded around $a = 0$.

2

$$T_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \frac{x^m}{m!}(\pm 1)$$

$m =$ last even # $\leq n$

b. Find a reasonable upper bound on the error in $T_n(x)$ at $x = 25$ and estimate how big n needs to be for the error to be less than 10^{-3}.

2

may use $(\ln(x!) \cong x\ln x - x)$ still dead 6 error!

$$|R_n(x)| = \left|\frac{f^{n+1}(\xi)}{(n+1)!} 25^{n+1}\right| \leq \frac{25^{n+1}}{(n+1)!} \leq 10^{-3}$$

$\Rightarrow n \approx 71$

c. Do you expect to have significant problems with roundoff error in calculating $T_n(25)$, with n as in part b? What if you calculate $T_n(1)$ with the same n?

1

yes $\left(1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \cdots \frac{x^{68}}{68!}\right) - \left(\frac{x^2}{2} + \frac{x^6}{6!} + \frac{x^{10}}{10!} + \cdots \frac{x^{70}}{70!}\right)$

$= \text{big} - \text{big} = 0.9912$

no, $T_n(1)$

2. Show that the iteration $x_{n+1} = x_n - \frac{f(x_n)}{q(x_n)}$ converges quadratically (at least) to the root r of $f(x) = 0$, if $lim_{x \to r} q(x) = f'(r) \neq 0$.

$$g(x) = x - \frac{f(x)}{q(x)}$$

3

$$g'(x) = 1 - \frac{f'(x)}{q(x)} + f(x)\frac{q'(x)}{q(x)^2}$$

$$\lim_{x \to r} g'(x) = 1 - \frac{f'(r)}{f'(r)} + \overset{0}{f(r)}\frac{q'(r)}{\cancel{f}(r)} \approx 0$$

3. For a certain root finder (Muller's method) it can be shown that $lim_{n\to\infty}\frac{e_{n+1}}{e_n e_{n-1} e_{n-2}} = M(\neq 0, \neq \infty)$. To estimate the order α of this method, assume $e_{n+1} = Ce_n^\alpha$, and $e_{n+1} = Me_n e_{n-1} e_{n-2}$. Find an equation satisfied by α, you need not actually find α.

$$e_n = Ce_{n-1}^\alpha = C(Ce_{n-2}^\alpha)^\alpha = C^{1+\alpha} e_{n-2}^{\alpha^2} \qquad \alpha = 1.839$$

3

$$e_{n-1} = \left(\frac{e_n}{C}\right)^{\frac{1}{\alpha}} \qquad e_{n-2} = \left(\frac{e_n}{C^{1+\alpha}}\right)^{\frac{1}{\alpha^2}} \qquad 1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} = \alpha$$

$$e_{n+1} = Me_n\left(\frac{e_n}{C}\right)^{\frac{1}{\alpha}}\left(\frac{e_n}{C^{1+\alpha}}\right)^{\frac{1}{\alpha^2}} = K e_n^{1+\frac{1}{\alpha}+\frac{1}{\alpha^2}}$$

4. To minimize the function $f(x, y) = 100(x^2 - y)^2 + (1 - x)^2$, set f_x and f_y equal to zero, and do one iteration of Newton's method, starting from $(1, 0)$ to solve this system of two equations and two unknowns. From $(1, 0)$, what is the direction of steepest descent?

$$f = 100x^4 - 200x^2y + 100y^2 + 1 - 2x + x^2$$

$$f_x = 400x^3 - 400xy + 2x - 2 = 0$$

3

$$f_y = -200x^2 + 200y = 0$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} 1200x_n^2 - 400y_n + 2 & -400x_n \\ -400x_n & 200 \end{pmatrix}^{-1} \begin{pmatrix} 400 \\ -200 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1202 & -400 \\ -400 & 200 \end{pmatrix}^{-1} \begin{pmatrix} 400 \\ -200 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

steepest descent $= -\nabla f = (-400, 200)$ or $(-2, 1)$

5. Explain how Newton's method could be used to compute A/B on a computer which only can add, subtract and multiply, but not divide.

$$f(x) = \frac{1}{x} - B$$

3

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - B}{\frac{-1}{x_n^2}} = 2x_n - Bx_n^2 \quad \text{iterate to find } x = \frac{1}{B}$$

then multiply $A\left(\frac{1}{B}\right) = \frac{A}{B}$

6. If $f(x) = (x-r)^m$, show that the "modified" Newton's method $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$ will converge in a single iteration to the root r, regardless of the starting value x_0. What would you predict would happen if this modified Newton method were applied to a more general function with a root of multiplicity m at r, that is to $f(x) = (x-r)^m h(x)$, where $h(r) \neq 0$? You can analyze the iteration using the techniques of section 3.4, or you can guess; but if you guess, it must be correct!

3

$$x_{n+1} = x_n - m\frac{(x_n-r)^m}{m(x_n-r)^{m-1}} = x_n - (x_n - r) = r$$

will converge quadratically near root

$$x_{n+1} = x_n - m\frac{(x_n-r)^m h(x_n)}{(x_n-r)^m h'(x_n) + m(x_n-r)^{m-1}h(x_n)}$$

$$x_{n+1} = x_n - m\frac{(x_n-r)h(x_n)}{(x_n-r)h'(x_n) + m\,h(x_n)}$$

$$g'(r) = 1 - m\frac{h(r)}{m(h(r))} = 1-1 = 0 \Rightarrow \text{quad conv}$$

$$g(x) = x - m(x-r)\left[\frac{h(x)}{(x-r)h'(x) + m\,h(x)}\right]$$

$$g'(x) = 1 - m(x-r)\left[\quad\right]' - m\left[\frac{h(x)}{(x-r)h'(x)+m\,h(x)}\right]$$

Math 5329, Test I (c)

Name Key

8

1. a. Find $T_n(x)$, the Taylor series of degree n for the function $f(x) = cosh(x)$, expanded around $a = 0$. Assume n is even. (Hint: $\frac{d}{dx}cosh(x) = sinh(x)$, $\frac{d}{dx}sinh(x) = cosh(x)$, sinh(0)=0, cosh(0)=1)

2

$$T_n(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \frac{x^n}{n!}$$

b. Find $E_n(x)$, the error in $T_n(x)$, and find a reasonable upper bound on $E_n(10)$. You can use the fact that $sinh(x)$ is a monotone increasing function.

2

$$E_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!} x^{n+1} = \frac{\sinh(\xi)}{(n+1)!} x^{n+1}$$

$$|E_n(10)| \le \frac{\sinh(10)}{(n+1)!} 10^{n+1}$$

c. Estimate the number of terms n required for $T_n(10)$ to approximate $cosh(10)$ to an accuracy of 10^{-4}.

2

$$\frac{\sinh(10)}{(n+1)!} 10^{n+1} \approx 10^{-4} \qquad \frac{10^{n+1}}{(n+1)!} \approx 10^{-8} \qquad n \approx 40$$

d. Would you expect roundoff error to be a serious concern in computing $T_n(10)$ in part (c)? Why or why not?

1

no, all term positive

2. a. To find a maximum or minimum of a function F(x,y), in calculus we set both partial derivatives to 0 and solve the resulting system of two equations. Explicitly write out what Newton's method looks like when applied to this system, in terms of F and its derivatives.

2

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix}_n^{-1} \begin{pmatrix} F_x \\ F_y \end{pmatrix}_n$$

b. If F(x,y) is a quadratic polynomial ($F(x,y) = a + bx + cy + dx^2 + exy + fy^2$), what can you say about convergence of Newton's method?

2

converges in a single iteration

3. It can be shown that for Mueller's method, $e_{n+1} \approx M e_n e_{n-1} e_{n-2}$. If Mueller's method is order α, ie, $e_{n+1} \approx C e_n^{\alpha}$, find an equation satisfied by α. Then use any method we have studied to find a root of this equation. (Hint: First write e_{n-1} and e_{n-2} in terms of e_n.)

3

$$e_n = C e_{n-1}^{\alpha} \qquad e_{n-1} = \left(\frac{e_n}{C}\right)^{\frac{1}{\alpha}}$$

$$e_{n-1} = C e_{n-2}^{\alpha} \qquad e_{n-2} = \left(\frac{e_{n-1}}{C}\right)^{\frac{1}{\alpha}} = D e_n^{\frac{1}{\alpha^2}}$$

$$e_{n+1} = \frac{M}{E} e_n e_n^{\frac{1}{\alpha}} e_n^{\frac{1}{\alpha^2}} = \frac{M}{E} e_n^{1+\frac{1}{\alpha}+\frac{1}{\alpha^2}}$$

$$1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} = \alpha \qquad \alpha^3 - \alpha^2 - \alpha - 1 = 0 \qquad \alpha_{n+1} = (1 + \alpha_n + \alpha_n^2)^{\frac{1}{3}}$$

$$\alpha_n \to 1.839$$

4. About how many bisection iterations should be required to obtain an error less than ϵ, knowing that f(a) and f(b) have opposite signs?

2

$$\frac{b-a}{2^N} = \epsilon \qquad N = \frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln 2}$$

5. Estimate the order of convergence for:

4

a. Newton's method applied to $f(x) = (x-3)^3(x-4)$, starting near the root r=3. 1

b. Same as (a) but starting near the root r=4. 2

c. Same as (a) but using Secant method. 1

d. Same as (a) but using Secant method and starting near the root r=4. 1.618

e. A root finder which produces consecutive errors of $10^{-5}, 10^{-7}$ and 10^{-12}. 2.5

f. The iteration $x_{n+1} = g(x_n)$ if $r = g(r)$ and $g'(r) = g''(r) = 0$ but $g'''(r) \neq 0$, and you start near the root r. 3

g. The bisection method. 1

h. The method $x_{n+1} = x_n - f(x_n)/f'(x_0)$. 1

Math 5329, Test I (d)

Name ____Key____

1. a. Find $T_n(x)$, the Taylor series of degree n for the function $f(x) = ln(1+x)$, expanded around $a = 0$.
(Hint: $f^{(n)}(x) = (-1)^{n-1}(n-1)!/(1+x)^n$, for $n \geq 1$.)

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots \pm \frac{x^n}{n}(-1)^{n-1}$$

2

b. Find $E_n(x)$, the error in $T_n(x)$, and find a reasonable upper bound on $|E_n(1)|$.

$$|E_n(x)| = \left|\frac{(-1)^n n!}{(1+\xi)^{n+1}} \frac{x^{n+1}}{(n+1)!}\right| = \left|\frac{1}{(1+\xi)^{n+1}} \frac{x^{n+1}}{n+1}\right| \leq \frac{1}{n+1}$$

2

c. Estimate the number of terms n required for $T_n(x)$ to approximate $f(1) = ln(2)$ to 5 decimal places accuracy.

$n \geq 100{,}000$

1

d. Would you expect roundoff error to be a serious concern in (c)? Why or why not?
(Hint: $1 + 1/2 + 1/3 + 1/4 + 1/5 + ... + 1/n \approx ln(n)$, for large n.)

1

no

$$\ln 2 \approx \left[1 + \frac{1}{3} + \frac{1}{5} + \dots \frac{1}{99999}\right] - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots \frac{1}{100000}\right]$$

$\approx$ pos $-$ neg

pos $\leq \ln(100000) \approx 11$

neg ≤ 11

$\ln 2 \approx 0.69$

7% difference or so

NOT "nearly equal"

2. Estimate the order of convergence of a root-finder that has consecutive errors $0.2, 0.08, 0.002048$.

2

$$.08 = M(.2)^{\alpha}$$
$$.002048 = M(.08)^{\alpha}$$

$$39.0625 \approx 2.5^{\alpha}$$
$$\alpha = 4$$

3. A certain computer stores floating point numbers in a 128-bit word. If a floating point number is written in normalized binary form ($1.xxxxx..._2 * 2^e$), it is stored using one sign bit (0 if the number is positive), then $e + 4095$ is stored in binary in the next 13 bits, and then the mantissa xxxxx... is stored in the final 114 bits. Show exactly how the number -27.125 would be stored on this computer. Also: approximately how many **decimal** digits of accuracy does this machine have?

3

$$-27.125 = 11011.001 = 1.1011001 \cdot 2^4$$

$$e + 4095 = 4099 = 1000000000011$$

Let: $1|1000000000011|1011001\,00...$

C00EC80...0 $\quad 2^{-114} \approx 5 \cdot 10^{-35}$ so ≈ 35 decimal digits

4. The fixed point iteration $x_{n+1} = x_n + sin(x_n)$ has roots at $r = n\pi$ for any integer n. Will this iteration converge if you start very close to the root $r = 0$? Will it converge if you start near the root $r = \pi$? In both cases, if it does converge, what is the order of convergence?

3

$$g(x) = x + \sin(x)$$
$$g'(x) = 1 + \cos(x)$$
$$g''(x) = -\sin(x)$$
$$g'''(x) = -\cos(x)$$

$r = 0$

$g'(0) = 2$ diverges

$r = \pi$

$$g'(\pi) = 0$$
$$g''(\pi) = 0$$
$$g'''(\pi) = -1$$

converges order 3

5. Write the secant iteration for solving $f(x) = 1/x - b = 0$, in a form where no divisions are required (thus this iteration could be used to compute $1/b$ on a computer which cannot do divisions).

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

3

$$= x_n - \frac{\left(\frac{1}{x_n} - b\right)(x_n - x_{n-1})}{\left(\frac{1}{x_n} - \frac{1}{x_{n-1}}\right)} = x_n + x_{n-1} - b x_n x_{n-1}$$

6. To minimize the function $f(x, y) = 10(2x + y)^2 + (x - 2)^2$, set f_x and f_y equal to zero, and do one iteration of Newton's method, starting from $(0, 1)$ to solve this system of two equations and two unknowns. The true minimum is obvious from looking at the function, where is the minimum? From $(0, 1)$, what is the direction of steepest descent? Which converges faster, Newton's method or the method of steepest descent?

3

$$f_x = 20(2x+y)2 + 2(x-2) = 82x + 40y - 4$$

$$f_y = 20(2x+y) = 40x + 20y$$

minimum

$$\nabla f(0,1) = \langle 36, 20 \rangle$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 82 & 40 \\ 40 & 20 \end{pmatrix}^{-1} \begin{pmatrix} 36 \\ 20 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

dir steepest descent $= -\nabla f = \langle -36, -20 \rangle$

Newton converges faster
(one iteration!)