Math 5329, Test II (a)

Name Key

1. a. Find the LU decomposition (no pivoting necessary) for

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

5

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & -\frac{4}{15} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 4 & -1 & 0 \\ 0 & \frac{15}{4} & -1 \\ 0 & 0 & \frac{56}{15} \end{bmatrix}$$

b. Solve $Ax = b$, where $b = (7, 1, 3)$ by first solving $Ly = b$, then $Ux = y$.

$y = (7, \frac{11}{4}, \frac{56}{15})$ $x = (2, 1, 1)$

2. If $A = L + D + U$ (L = strictly lower triangular, U is strictly upper triangular and D is diagonal), what is the matrix whose eigenvalue must be less than one in absolute value for convergence, for the

3

a. Jacobi Iterative Method $-D^{-1}(L+U)$

b. Gauss-Seidel Iterative Method $-(L+D)^{-1}U$

c. SOR Iterative Method $\left(L + \frac{D}{\omega}\right)^{-1}\left(\frac{D}{\omega} - D - U\right)$

4

3. Do several iterations of the inverse power method to find the smallest eigenvalue (in absolute value) of A, and the corresponding eigenvector, if $\approx \begin{pmatrix} -0.72 \\ 1 \end{pmatrix}$

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad x_2 = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$

$x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad x_3 = \begin{pmatrix} 31 \\ -43 \end{pmatrix}$ $\quad x_3 \cong -6.2\, x_2 \quad \lambda = \frac{-1}{6.2} = -0.16$

$\vec{x}_{min} = (31, -43)$

4

4. If $p_N(x)$ is the polynomial of degree N which interpolates $f(x) = cos(3x)$ at $N+1$ uniformly spaced points between 0 and π, find a bound, involving only N, on $max(0 \leq x \leq \pi)|p_N(x) - f(x)|$ Will your bound go to zero as $N \to \infty$? yes

$$|p_N(x) - f(x)| = \left| \frac{(x-x_0)\cdots(x-x_N)}{(N+1)!}\, 3^{N+1} \cos(3\xi) \text{ or } \sin(3\xi) \right| \leq \frac{\pi^{N+1} 3^{N+1}}{(N+1)!}$$

$$= \frac{(3\pi)^{N+1}}{(N+1)!} \to 0 \quad \text{as} \quad N \to \infty$$

4

5. Determine the equations which must be satisfied for

$$s(x) = \begin{cases} a(x-2)^2 + b(x-1)^3 & x \leq 1 \\ c(x-2)^2 & 1 \leq x \leq 3 \\ d(x-2)^2 + e(x-3)^3 & 3 \leq x \end{cases}$$

to be a cubic spline.

		$x=1$		$x=3$	
S	$a(x-2)^2 + b(x-1)^3$	$a=c$	$c(x-2)^2$	$c=d$	$d(x-2)^2 + e(x-3)^3$
S'	$2a(x-2) + 3b(x-1)^2$	$-2a=-2c$	$2c(x-2)$	$2c=2d$	$2d(x-2) + 3e(x-3)^2$
S''	$2a + 6b(x-1)$	$2a=2c$	$2c$	$2c=2d$	$2d + 6e(x-3)$

$a=c$ $\qquad c=d$

$a = c = d$

Math 5329, Test II (b)

Name ____Key________________

1. If

$$A = \begin{bmatrix} 1 & a \\ a & 2 \end{bmatrix}$$

Find the range of values of a for which the Gauss-Seidel method will converge, when applied to a system with matrix A.

4

$$-(L+D)^{-1}U = \begin{bmatrix} 0 & -a \\ 0 & \frac{a^2}{2} \end{bmatrix} \qquad \lambda = 0, \frac{a^2}{2}$$

$$-\sqrt{2} < a < \sqrt{2}$$

2. a. Find the LU decomposition (without pivoting) of

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & \frac{61}{16} \end{bmatrix}$$

6

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{3}{8} & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 0 & 4 & 3/2 \\ 0 & 0 & 3 \end{bmatrix}$$

b. Now find the Choleski decomposition LL^T of A (not necessarily the same L as part (a)).

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ \frac{1}{2} & \frac{3}{4} & \sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & \frac{1}{2} \\ 0 & 2 & 3/4 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

c. Prove that A is positive definite. (Hint: Use part (b))

$$x^TAx = x^TLL^Tx = (L^Tx)^TL^Tx = \|L^Tx\|^2 \geq 0$$

$= 0$ only if $L^Tx = 0 \Rightarrow x = 0$ (since L^{-1} exists)

3. Do several iterations of the inverse power method to find the smallest eigenvalue (in absolute value) of A, and the corresponding eigenvector, if

$$A = \begin{bmatrix} -4/6 & 2/6 \\ 5/6 & -1/6 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

4

$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $x_1 = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$ $x_2 = \begin{pmatrix} 21 \\ 51 \end{pmatrix}$ $x_3 = \begin{pmatrix} 123 \\ 309 \end{pmatrix}$ $x_3 \approx 6x_2$

$\lambda_{min} = \frac{1}{6}$ $x \approx \begin{pmatrix} 0.4 \\ 1 \end{pmatrix}$

4. If $p_3(x)$ is the polynomial of degree 3 which interpolates $f(x) = ln(x)$ at $x = 1.0, 1.1, 1.2, 1.3$, find as small a bound as possible on $max_{1.1 \le x \le 1.2}|p_3(x) - f(x)|$. (Note: the range is only (1.1, 1.2)).

assume $\xi = 1$, $x = 1.15$

4

$$\left|\frac{f^{IV}(\xi)}{4!}(x-1)(x-1.1)(x-1.2)(x-1.3)\right| = \left|\frac{6}{\xi^4}\frac{1}{24}(x-1)(x-1.1)(x-1.2)(x-1.3)\right|$$

$$\le \frac{1}{4}(.15)^2(.05)^2 = 1.4 \cdot 10^{-5}$$

5. Set up (don't try to solve) the equations to determine the cubic spline

$$\begin{aligned} s(x) &= a + bx + cx^2 + dx^3 \quad -1 \le x \le 0 \\ & e + fx + gx^2 + hx^3 \quad 0 \le x \le 1 \end{aligned}$$

4

which interpolates to $f(x) = sin(\frac{\pi}{2}x)$ at $x = -1, 0, 1$ and which matches $f''(x)$ at the endpoints $x = -1$ and $x = 1$.

$f(-1) = -1$ $\quad f''(x) = -\frac{\pi^2}{4} \sin\left(\frac{\pi}{2}x\right)$

$f(0) = 0$ $\quad f''(-1) = \frac{\pi^2}{4}$

$f(1) = 1$ $\quad f''(1) = -\frac{\pi^2}{4}$

$a - b + c - d = -1$

$a = 0$

$e = 0$

$e + f + g + h = 1$

$2c - 6d = \frac{\pi^2}{4}$

$2g + 6h = \frac{-\pi^2}{4}$

$b = f$

$c = g$

Math 5329, Test II (c)

Name Key

1. a. Find the LU decomposition (without pivoting) for

$$A = \begin{bmatrix} 4 & -8 \\ -8 & 25 \end{bmatrix}$$

3

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 & -8 \\ 0 & 9 \end{bmatrix}$$

b. Now find the Cholesky decomposition LL^T of the positive definite matrix A. (Hint: First, write LU in the form LDU_2), where D is the diagonal of U, and $U_2 = D^{-1}U$ and note that $U_2 = L^T$.)

3

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 0 & 3 \end{bmatrix}$$

2. Find the condition number (using the L_∞ norm) of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 + 10^{-10} \end{bmatrix}$$

3

If machine precision is about 10^{-16} about how many significant digits should we expect when $Ax = b$ is solved, using Gaussian elimination with partial pivoting?

$$A^{-1} = \frac{\begin{bmatrix} 4+10^{-10} & -2 \\ -2 & 1 \end{bmatrix}}{10^{-10}}$$

$$\text{cond}(A) = \|A\| \, \|A^{-1}\| \cong 6 \cdot 6 \cdot 10^{10}$$

$$= 3.6 \cdot 10^{11}$$

about 5 digits left

3. Do several iterations of the inverse power method to find the smallest eigenvalue (in absolute value) of A, and the corresponding eigenvector, if

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

3

$$A^{-1} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} -2 \\ 6 \end{pmatrix} \to \begin{pmatrix} -44 \\ 84 \end{pmatrix} \to \begin{pmatrix} -680 \\ 1272 \end{pmatrix}$$

15.5

15.1

$$\lambda(A) \approx \frac{1}{15} \; (0.0657) \qquad \vec{x} \approx \begin{pmatrix} 1 \\ -1.9 \end{pmatrix}$$

4. If $p_2(x)$ is the polynomial of degree 2 which interpolates $f(x) = ln(x)$ at $x = 0.5, 1.0, 1.5$, find reasonable upper *and* lower bounds on $|p_2(2) - ln(2)|$.

4

$$\left| \frac{f'''(\xi)}{3!} (x-\tfrac{1}{2})(x-1)(x-1.5) \right| = \left| \frac{2}{\xi^3} \frac{1}{6} (2-\tfrac{1}{2})(2-1)(2-1.5) \right|$$

$$= \frac{0.25}{\xi^3} \qquad \frac{0.25}{(.5)^3} \geq \text{error} \geq \frac{0.25}{2^3}$$

$$\frac{1}{2} \leq \xi \leq 2 \qquad 2 \geq \text{error} \geq \frac{1}{32}$$

5. Determine a,b,c,d,e so that

$$s(x) = \begin{cases} a(x-2)^2 + b(x-1)^3 & x \le 1 \\ c(x-2)^2 & 1 \le x \le 3 \\ d(x-2)^2 + e(x-3)^3 & 3 \le x \end{cases}$$

4

is the cubic spline interpolant through the points (0,2),(1,4),(3,4) and (4,2). (There is a unique solution, even though no end conditions are specified.)

		$x=1$		$x=3$	
s	$a(x-2)^2+b(x-1)^3$	$a=c$	$c(x-2)^2$	$c=d$	$d(x-2)^2+e(x-3)^3$
s'	$2a(x-2)+3b(x-1)^2$	$a=c$	$2c(x-2)$	$c=d$	$2d(x-2)+3e(x-3)^2$
s''	$2a+6b(x-1)$	$a=c$	$2c$	$c=d$	$2d+6e(x-3)$
		$=4$		$=4$	

$a=c=d=4$

$s(0) = 4a - b$

$= 16 - b = 2$

$b = 14$

$s(4) = 4d + e$

$= 16 + e = 2$

$e = -14$

Math 5329, Test II (Q)

Name _____Key_____

1. a. Find the LU decomposition (no pivoting necessary) for

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$$

2

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix} = A$$

b. What is an LU decomposition good for?

1

cuts work from $O(N^3)$ to $O(N^2)$ when solving additional system with same matrix

c. Find the Cholesky decomposition LL^T of A.

2

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

d. Show that a matrix that has a Cholesky decomposition $A = LL^T$, where L nonsingular, is positive definite.

2

$$x^T A x = x^T L L^T x = (L^T x)^T L^T x$$

$$= \|L^T x\|^2 > 0 \quad \text{unless } L^T x = 0$$

$$\Rightarrow x = 0$$

since L nonsingular

2. If $A = L + D + U$ (L = strictly lower triangular, U is strictly upper triangular and D is diagonal), what is the matrix whose eigenvalues must be less than one in absolute value for convergence, for the

2

   a. Jacobi Iterative Method

$$-D^{-1}(L+U)$$

   b. Gauss-Seidel Iterative Method

$$-(L+D)^{-1}U$$

3. To find the eigenvalue of A closest to a number p, the power method can be applied to what matrix? Explain how an LU decomposition could be used to make this iteration more efficient.

2

$$(A-pI)^{-1}$$

solve $(A-pI)x_{n+1} = x_n$ repeatedly, we LU decomposition on second, etc iteration.

4. If $p_N(x)$ is the polynomial of degree N which interpolates $f(x) = e^{3x}$ at $N+1$ uniformly spaced points between 0 and 10, find a bound, involving only N, on $max(0 \le x \le 10)|p_N(x) - f(x)|$ Will your bound go to zero as $N \to \infty$?

3

$$|p_N(x)-f(x)| = \left|\frac{f^{N+1}(\xi)}{(N+1)!}(x-x_0)\cdots(x-x_N)\right| \le \frac{3^{N+1}e^{3\xi}}{(N+1)!}10^{N+1}$$

$$\le \frac{30^{N+1}e^{30}}{(N+1)!}$$

yes $\to 0$ as $N \to \infty$

5. What is the condition number (using the L_∞ norm) of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 + 10^{-9} \end{bmatrix}$$

If our computer has about 20 decimal digits precision, about how many significant decimal digits would we expect in the solution of $Ax = b$?

2

$$A^{-1} = \frac{\begin{bmatrix} 1+10^{-9} & -1 \\ -1 & 1 \end{bmatrix}}{10^{-9}} \qquad \|A\|_\infty = 2 \qquad \|A^{-1}\| \doteq 2 \cdot 10^{9}$$

cond $(A) = 4 \cdot 10^{9}$

about 11 digits

6. A quintic spline interpolant is a function which is a polynomial of degree five or less in each interval $(x_{i-1}, x_i), i = 1, ..., N$ and passes through the points $(x_i, y_i), i = 0, ..., N$ and is continuous and has continuous first, second, third and fourth derivatives.

4

a. How many unknown coefficients need to be determined? (Hint: There are N intervals and the quintic has how many coefficients in each?)

$6N$

b. How many interpolation conditions are there? (Hint: There are two interpolation conditions for each interval.)

$2N$

c. How many continuity conditions are there? (Hint: s(x) is automatically continuous because of the interpolation conditions, so we only need to require that s', s'', s''', s^{iv} be continuous at each *interior* point—how many interior points are there?)

$4(N-1)$

d. If you add the number of interpolation conditions (part b) and continuity conditions (part c), does this equal the number of unknowns (part a)? If not, what needs to be done to make the quintic spline interpolation problem have a unique solution?

no, need 4 end conditions