Math 5329, Test III (a)

Name ____Key____

1. a. Find r, s which make the quadrature formula below as high order as possible:

$\int_a^b f(x)dx \approx \sum_{i=1}^N \frac{h}{3}[f(x_{i-1}+rh) + f(x_{i-1}+\frac{1}{2}h) + f(x_{i-1}+sh)]$

(Hint: how are r and s related, by symmetry?)

3

$r = 0.14644$
$s = 0.85355$

$\int_0^h f(x)\,dx \approx \frac{h}{3}f(rh) + \frac{h}{3}f(\frac{1}{2}h) + \frac{h}{3}f((1-r)h)$

$f=1 \quad h \overset{\checkmark}{=} \frac{h}{3}+\frac{h}{3}+\frac{h}{3} = h$

$=x \quad \frac{1}{2}h^2 \overset{\checkmark}{=} \frac{h}{3}rh + \frac{h}{3}\frac{h}{2} + \frac{h}{3}(1-r)h = \frac{1}{2}h^2$

$=x^2 \quad \frac{1}{3}h^3 = \frac{h}{3}r^2h^2 + \frac{h}{3}\frac{h^2}{4} + \frac{h}{3}(1-r)^2h^2 = \frac{h^3}{12}[8r^2-8r+5]$

$=x^3 \quad \frac{1}{4}h^4 = \frac{h}{3}r^3h^3 + \frac{h}{3}\frac{h^3}{8} + \frac{h}{3}(1-r)^3h^3 = h^4[\frac{1}{24}+\frac{1}{3}-r+r^2]$

$8r^2-8r+5=4$

$\binom{r}{s} = \frac{1}{2} \pm \sqrt{\frac{1}{8}}$

$\rightarrow r^2 - r + \frac{1}{8} \overset{\checkmark}{=} 0$

b. With this choice for r, s, what is the global order of this rule?

1

$O(h^4)$

2. a. Is the method $3U_{n+1} - 4U_n + U_{n-1} = 2hf(t_{n+1}, U_{n+1})$ (for approximating $u' = f(t,u)$) stable?

$3\lambda^2 - 4\lambda + 1 = 0$

2

$(3\lambda - 1)(\lambda - 1) = 0$

$\lambda = \frac{1}{3}, 1$ so stable

b. Is it explicit or implicit?

implicit

1

c. Calculate the truncation error. (Hint: put in normalized form first.)

$$T = \frac{3u(t+h) - 4u(t) + u(t-h)}{2h} - u'(t+h)$$

$$= \frac{3\left(u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' + \dots\right) - 4u + u - hu' + \frac{h^2}{2}u'' - \frac{h^3}{6}u''' \dots}{2h}$$

3

$$-\left[u' + u''h + u'''\frac{h^2}{2} + \dots\right] = \frac{2hu' + 2h^2u'' + \frac{1}{3}h^3u'''}{2h} - \left[u' + u''h + u'''\frac{h^2}{2} \dots\right]$$

$$= -\frac{1}{3}h^2u''' + \dots$$

3. A certain quadrature method gives the following estimates of an integral:

h	I_h
0.125	42.0642089572
0.0625	42.0699513233
0.03125	42.0703214561

3

Estimate the order of convergence (without knowing the true value of the integral).

$$2^\alpha = \frac{I_1 - I_2}{I_2 - I_3} = \frac{-0.0057423}{-0.0003701} = 15.5$$

$$\alpha = 3.96$$

4. Estimate u(1.1) by taking one step of the Taylor series of order three (involving up to third derivatives in the Taylor series), with h=0.1, for the problem $\frac{du}{dt} = -tu, u(1) = 2$.

3

$$u(1+h) \doteq u(1) + u'(1)h + u''(1)\frac{h^2}{2} + u'''(1)\frac{h^3}{6} = 2 - 2h + \frac{4}{6}h^3$$

$$\doteq 1.8006666$$

$u' = -tu$ $\quad u'(1) = -(1)(2) = -2$

$u'' = -u - tu'$ $\quad u''(1) = -2 - (1)(-2) = 0$

$u''' = -u' - u' - tu''$ $\quad u'''(1) = -2(-2) - 1(0) = 4$

5. Write the third order equation:

$$u''' - sin(u'') + e^t u' + 2t\ cos(u) = 25$$
$$u(0) = 5, u'(0) = 3, u''(0) = 7$$

as a system of three first order equations:

3

$$u_1' = f_1(t, u_1, u_2, u_3)$$
$$u_2' = f_2(t, u_1, u_2, u_3)$$
$$u_3' = f_3(t, u_1, u_2, u_3)$$

with

$$u_1(0) = A$$
$$u_2(0) = B$$
$$u_3(0) = C$$

That is, find f_1, f_2, f_3, A, B, C.

$$u_1' = u_2$$
$$u_2' = u_3$$
$$u_3' = \sin(u_3) - e^t u_2 - 2t\cos(u_1) + 25$$

$$u_1(0) = 5$$
$$u_2(0) = 3$$
$$u_3(0) = 7$$

Math 5329, Test III (B)

Name Key

1. Determine the (global) order of the quadrature rule:
$\int_a^b f(x)dx \approx \sum_{i=1}^{N}\left[\frac{h}{4}f(x_{i-1})+\frac{3h}{4}f(x_{i-1}+\frac{2h}{3})\right]$

$$\int_0^h f(x)\,dx \doteq \frac{h}{4}f(0)+\frac{3h}{4}f\left(\frac{2h}{3}\right)$$

4

$f=1$	$h =$	$\frac{h}{4}+\frac{3h}{4}=h$ ✓
x	$\frac{1}{2}h^2 =$	$\frac{3h}{4}\left(\frac{2h}{3}\right)=\frac{1}{2}h^2$ ✓
x^2	$\frac{1}{3}h^3 =$	$\frac{3h}{4}\left(\frac{2h}{3}\right)^2=\frac{1}{3}h^3$ ✓ $\rightarrow O(h^3)$
x^3	$\frac{1}{4}h^4 =$	$\frac{3h}{4}\left(\frac{2h}{3}\right)^3=\frac{2}{9}h^4$ no

2. Find A, B, C such that the approximation $u'(t)\approx\frac{Au(t)+Bu(t-h)+Cu(t-2h)}{h}$ is as high order as possible.

$$\frac{Au+B\left(u-hu'+\frac{h^2}{2}u''+\dots\right)+C\left(u-2hu'+\frac{4h^2}{2}u''+\dots\right)}{h}$$

$$=\frac{(A+B+C)u+hu'(-B-2C)+\frac{h^2u''}{2}(B+4C)\dots}{h}=u'$$

4

$A+B+C=0$
$-B-2C=1$
$B+4C=0$

$C=\frac{1}{2}$
$B=-2$
$A=\frac{3}{2}$

3. If the second order Taylor series method (one more term than Euler's method) is used to solve $u' = t^2\sqrt{1+u^2}$, write u_{n+1} in terms of h, t_n and u_n only. ($t_n = nh, u_n \approx u(t_n)$)

$$u'' = t^2 \tfrac{1}{2}(1+u^2)^{-\frac{1}{2}} 2uu' + 2t(1+u^2)^{\frac{1}{2}}$$

$$= \frac{t^2 u}{\sqrt{1+u^2}} t^2\sqrt{1+u^2} + 2t\sqrt{1+u^2} = t^4 u + 2t\sqrt{1+u^2}$$

4

$$u_{n+1} = u_n + h\, t_n^2\sqrt{1+u_n^2} + \frac{h^2}{2}\left(t_n^4 u_n + 2t_n\sqrt{1+u_n^2}\right)$$

4. Reduce

$$y'' = 3y'y - e^t z$$
$$z'' = z'z - \sqrt{y}$$

to a system of 4 first order equations.

3

$u_1 \equiv y$
$u_2 \equiv y'$
$u_3 \equiv z$
$u_4 \equiv z'$

$$u_1' = u_2$$
$$u_2' = 3u_2u_1 - e^t u_3$$
$$u_3' = u_4$$
$$u_4' = u_4 u_3 - \sqrt{u_1}$$

5. a. Is the method $11U_{n+1} - 18U_n + 9U_{n-1} - 2U_{n-2} = 6hf(t_{n+1}, U_{n+1})$ (for approximating $u' = f(t,u)$) stable?

$$11\lambda^3 - 18\lambda^2 + 9\lambda - 2 = 0$$

$$(\lambda-1)(11\lambda^2 - 7\lambda + 2) = 0$$

2

$$\lambda_1 = 1 \qquad |\lambda_2| = |\lambda_3| = 0.426$$

$$\lambda_{2,3} = \frac{7 \pm \sqrt{39}\, i}{22}$$

yes

b. Is it explicit or implicit?

1

implicit

c. Is it consistent? (Extra credit: what is the truncation error?)

$$T = \frac{11u(t+h) - 18u(t) + 9u(t-h) - 2u(t-2h)}{6h} - u'(t+h)$$

2

yes

+2ec

$$= \Big[11\left(u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' + \frac{h^4}{24}u^{iv} + \dots\right)$$

$$-18u$$

$$+9\left(u - hu' + \frac{h^2}{2}u'' - \frac{h^3}{6}u''' + \frac{h^4}{24}u^{iv} + \dots\right)$$

$$-2\left(u - 2hu' + \frac{4h^2}{2}u'' - \frac{8h^3}{6}u''' + \frac{16h^4}{24}u^{iv} \dots\right)\Big] / 6h$$

$$-\left(u' + hu'' + \frac{h^2}{2}u''' + \frac{h^3}{6}u^{iv} \dots\right) = -\frac{h^3 u^{iv}}{4}$$

Math 5329, Test III (c)

Name Key

1. a. Find r, s which make the quadrature formula below as high order as possible ($x_i = a + ih, h = (b-a)/N$):

$$\int_a^b f(x)dx \approx \sum_{i=1}^N \frac{h}{2}[f(x_{i-1}+rh) + f(x_{i-1}+sh)]$$

4

(Hint: how are r and s related, by symmetry?)

$$\int_0^h f(x)\,dx \cong \frac{h}{2} f(rh) + \frac{h}{2} f((1-r)h)$$

$f=1$ $\quad h = \int_0^h 1\,dx \overset{\checkmark}{\underset{?}{=}} \frac{h}{2} + \frac{h}{2} = h$

$f=x$ $\quad \frac{h^2}{2} = \int_0^h x\,dx \overset{\checkmark}{\underset{?}{=}} \frac{h}{2} rh + \frac{h}{2}(1-r)h = \frac{h^2}{2}$

$f=x^2$ $\quad \frac{h^3}{3} = \int_0^h x^2\,dx = \frac{h}{2} r^2h^2 + \frac{h}{2}(1-r)^2h^2 = h^3(r^2 - r + \frac{1}{2})$

$f=x^3$ $\quad \frac{h^4}{4} = \int_0^h x^3\,dx = \frac{h}{2} r^3h^3 + \frac{h}{2}(1-r)^3h^3 = \frac{h^4}{2}(3r^2 - 3r + 1)$

b. With this choice for r, s, what is the global order of this rule?

2

$r^2 - r + \frac{1}{2} = \frac{1}{3}$ $\qquad 3r^2 - 3r + 1 = \frac{1}{2}$

$r^2 - r + \frac{1}{6} = 0$ $\qquad r^2 - r + \frac{1}{6} = 0$

$$r = \frac{1 \pm \sqrt{1 - \frac{4}{6}}}{2} = \frac{1}{2} \pm \sqrt{\frac{1}{12}} = r$$

$s = 1 - r$

$d.o.p = 3 \Rightarrow O(h^4)$

2. a. Is the following method stable? (Justify answer)

$$\frac{U_{k+1}-U_{k-2}}{3h} = \frac{1}{2}f(t_k, U_k) + \frac{1}{2}f(t_{k-1}, U_{k-1})$$

$r^3 - 1 = 0$

3

all $|r| = 1$ so stable $\quad r = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

b. (Extra credit) Find the truncation error, and tell if the method is consistent or not.

$$T = \frac{u(t+h) - u(t-2h)}{3h} - \frac{1}{2}u'(t) - \frac{1}{2}u'(t-h)$$

+3

$= \frac{1}{4}h^2 u''' + \ldots \quad$ yes consistent

3. a. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-11} when $h = 10^{-4}$. Estimate the order of the method.

2

$10^{-5} = M(10^{-2})^\alpha$

$10^{-11} = M(10^{-4})^\alpha$

$10^6 = (10^2)^\alpha \quad \alpha = 3$

b. A differential equation solver gives an answer $u(1) = 1.020$ when $h = 0.1$, and $u(1) = 1.004$ when $h = 0.05$, and $u(1) = 1.003$ when $h = 0.025$. Estimate the order of the method.

2

$1.020 - T = M(.1)^\alpha$

$1.004 - T = M(.05)^\alpha$

$1.003 - T = M(.025)^\alpha$

$.016 = M(.1^\alpha - .05^\alpha)$

$.001 = M(.05^\alpha - .025^\alpha)$

$16 = 2^\alpha \quad \alpha = 4$

4. a. Write the third order differential equation $u''' - 3u'' - u = t^2$ as a system of three first order equations, that is, in the form:

$u' = f(t, u, v, w) =$ V

2

$v' = g(t, u, v, w) =$ W

$w' = h(t, u, v, w) =$ $3w + u + t^2$

b. Now write out the formulas for $u_{n+1}, v_{n+1}, w_{n+1}$ for Euler's method applied to this system of first order equations:

$u_{n+1} =$ $u_n + h\,v_n$

2

$v_{n+1} =$ $v_n + h\,w_n$

$w_{n+1} =$ $w_n + h\,(3w_n + u_n + t_n^2)$

5. If the third order Taylor series method (two more terms than Euler's method) is used to solve $u' = t^2 + 5u$, write u_{n+1} in terms of h, t_n and u_n only. ($t_n = nh, u_n \approx u(t_n)$)

$u' = t^2 + 5u$

3

$u'' = 2t + 5(t^2 + 5u) = 2t + 5t^2 + 25u$

$u''' = 2 + 10t + 25(t^2 + 5u) = 2 + 10t + 25t^2 + 125u$

$u_{n+1} = u_n + h(t_n^2 + 5u_n) + \frac{h^2}{2}(2t_n + 5t_n^2 + 25u_n)$
$+ \frac{h^3}{6}(2 + 10t_n + 25t_n^2 + 125u_n)$

Math 5329, Test III (2)

Name Key

(4)

1. Find A, B, C which make the quadrature formula below as high order as possible:

$\int_0^h f(x)dx \approx Ahf(0.2h) + Bhf(0.5h) + Chf(0.8h)$

$h = Ah + Bh + Ch$ $\qquad A+B+C=1$

$\frac{1}{2}h^2 = Ah\,0.2h + Bh\,0.5h + Ch\,0.8h$ $\qquad 0.2A + 0.5B + 0.8C = 0.5$

$\frac{1}{3}h^3 = Ah\,0.04h^2 + Bh\,0.25h^2 + Ch\,0.64h^2$ $\qquad 0.04A + 0.25B + 0.64C = \frac{1}{3}$

$A = C \Rightarrow$ $\qquad 2A+B=1$, $\quad A+0.5B=0.5$, $\quad 0.68A+0.25B=\frac{1}{3}$ $\quad\Rightarrow\quad A=C=\frac{25}{54}=0.46296$, $\quad B=\frac{4}{54}=0.07407$

(4)

2. Use Taylor series to find the error in the approximation:

$u'(t) \approx \frac{-u(t+2h)+8u(t+h)-8u(t-h)+u(t-2h)}{12h}$

$-1 \quad u(t+2h) = u + 2u'h + 4u''\frac{h^2}{2} + 8u'''\frac{h^3}{6} + 16u^{iv}\frac{h^4}{24} + 32u^{v}\frac{h^5}{120}$

$8 \quad u(t+h) = u + u'h + u''\frac{h^2}{2} + u'''\frac{h^3}{6} + u^{iv}\frac{h^4}{24} + u^{v}\frac{h^5}{120}$

$-8 \quad u(t-h) = u - u'h + u''\frac{h^2}{2} - u'''\frac{h^3}{6} + u^{iv}\frac{h^4}{24} - u^{v}\frac{h^5}{120}$

$1 \quad u(t-2h) = u - 2u'h + 4u''\frac{h^2}{2} - 8u'''\frac{h^3}{6} + 16u^{iv}\frac{h^4}{24} - 32u^{v}\frac{h^5}{120}$

$\text{num} = 12u'h \qquad -48u^{v}\frac{h^5}{120}$

$\text{error} = \frac{12u'h - 48u^{v}\frac{h^5}{120}}{12h} - u' = -u^{v}\frac{h^4}{30} + \ldots$

3. If the third order Taylor series method (two more terms than Euler's method) is used to solve $u' = t^2u^3$, write u_{n+1} in terms of h, t_n and u_n only. ($t_n = nh, u_n \approx u(t_n)$)

4

$u' = t^2u^3$

$u'' = 3t^4u^5 + 2tu^3$

$u''' = 15t^6u^7 + 18t^3u^5 + 2u^3$

$u_{n+1} = u_n + h\, t_n^2 u_n^3 + \frac{h^2}{2}\left(3t_n^4u_n^5 + 2t_nu_n^3\right) + \frac{h^3}{6}\left(15t_n^6u_n^7 + 18t_n^3u_n^5 + 2u_n^3\right)$

4. Write the third order equation:

$$u''' - cos(u'') + e^t u' + 4t\ sin(u) = 25$$

as a system of three first order equations, of the form:

2

$u_1' = f_1(t, u_1, u_2, u_3) =$ u_2
$u_2' = f_2(t, u_1, u_2, u_3) =$ u_3
$u_3' = f_3(t, u_1, u_2, u_3) =$ $\cos u_3 - e^t u_2 - 4t \sin(u_1) + 25$

5. a. Is the method $\frac{3}{2}U_{n+1} - 2U_n + \frac{1}{2}U_{n-1} = hf(t_{n+1}, U_{n+1})$ (for approximating $u' = f(t,u)$) stable?

2

$$\frac{3}{2}\lambda^2 - 2\lambda + \frac{1}{2} = 0 \qquad 3\lambda^2 - 4\lambda + 1 = 0$$

$$(3\lambda - 1)(\lambda - 1) = 0$$

$\lambda = 1, \frac{1}{3}$ so stable

1

b. Is it explicit or implicit? implicit

3

c. Calculate the truncation error. (Hint: put in normalized form first.)

$$T = \frac{\frac{3}{2}u(t+h) - 2u(t) + \frac{1}{2}u(t-h)}{h} - u'(t+h)$$

$$= \frac{\frac{3}{2}\left(u + u'h + u''\frac{h^2}{2} + u'''\frac{h^3}{6} \ldots\right) - 2u + \frac{1}{2}\left(u - u'h + u''\frac{h^2}{2} - u'''\frac{h^3}{6}\right)}{h}$$

$$- \left[u' + u''h + u'''\frac{h^2}{2} + \ldots\right] = \frac{u'h + u''h^2 + u'''\frac{h^3}{6} \ldots}{h}$$

$$- \left\{u' + u''h + u'''\frac{h^2}{2} + \ldots\right\} = -\frac{1}{3}h^2 u''' + \ldots$$