Math 5330, Test I

Name ______________________________

1. a. Show that any matrix which has a "Cholesky" decomposition $A = LL^T$, with L nonsingular, is positive definite, that is, show it is symmetric and $x^T Ax > 0$ for any nonzero vector x.

   b. Show that

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

   is positive definite, by finding its LU decomposition.

2. An N by N band matrix has $N^{1/3}$ non-zero diagonals below the main diagonal and the same number above. If N is large, approximately how many multiplications are done:

   a. during the forward elimination, if no pivoting is done?
   b. during the forward elimination, if partial pivoting is done?
   c. during back substitution, if no pivoting is done?
   d. during back substitution, if partial pivoting is done?

3. A MATLAB program which solves a symmetric linear system, with no pivoting, does most of its work in the loops:

```
for I=1:N-1
   for J=I+1:N
      for K=J:N
         A(J,K) = A(J,K) - LJI*A(I,K)
      end
   end
end
```

Approximately how many multiplications are done (show work)? How does this compare to Gaussian elimination for a nonsymmetric system?

4. a. If a matrix is decomposed into its (strictly) subdiagonal, diagonal, and (strictly) superdiagonal parts, $A = L + D + U$, the Jacobi iterative method for solving $Ax = b$ will converge if and only if all eigenvalues of what matrix are less than 1 in absolute value?

   b. Same question, for the Gauss-Seidel method.

   c. Using parts [a.] and [b.], show that both Jacobi and Gauss-Seidel methods will converge if A is either upper triangular or lower triangular, and all its diagonal elements are nonzero. (Hint: the eigenvalues of an upper or lower triangular matrix are its diagonal entries.)

5. Approximately how many significant digits would you expect in the solution of $Ax = b$ if Gaussian elimination with partial pivoting is used on a computer with machine precision $\epsilon = 10^{-12}$, and

$$A = \begin{bmatrix} 1.000001 & 1 \\ 1 & 1 \end{bmatrix}$$

6. Define:

   a. orthogonal matrix

   b. lower Hessenberg matrix

   c. permutation matrix

   d. $\|x\|_p$, if x is a vector and $1 \leq p < \infty$

   e. $\|A\|_p$, if A is a matrix