Math 5330, Test I

Name ____________________________

1. If

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

   find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $PA = LU$.

2. Prove that $\frac{\|\Delta x\|}{\|x\|} \leq cond(A)\frac{\|\Delta b\|}{\|b\|}$ if $Ax = b$ and $A(x + \Delta x) = b + \Delta b$.

3. If we use the usual finite difference approximation, the DE $u''(x) = f(x), u(0) = u(\pi) = 0$ becomes:

$$U_{i+1} - 2U_i + U_{i-1} = h^2 f(x_i), \quad i = 1, ..., N-1$$
$$U(x_0) = U(x_N) = 0$$

where $h = \pi/N, x_i = ih, U_i \approx u(x_i)$.

a. This is a linear system of $N-1$ equations for the $N-1$ unknowns $U_1, ..., U_{N-1}$. If a band solver is used to solve the system, the work is proportional to what power of N?

b. If Jacobi's iterative method is used to solve it, the iteration will take the form $U^{k+1} = BU^k + c$; what is the matrix B?

c. What is $\|B\|_\infty$?

d. What are the eigenvalues of the B matrix (hint: for any $m = 1, ..., N-1$, the vector U with components $U_i = sin(mx_i)$ is an eigenvector. You will need the trig identity $sin(a+b) = sin(a)cos(b) + cos(a)sin(b)$)

e. What is the largest eigenvalue of B in absolute value? Will the Jacobi method converge?

4. Which of the following linear systems has the largest condition number? Would you expect to have serious round-off error problems if you solved this system, using Gauss elimination with partial pivoting?

$$\begin{bmatrix} 1 & 10^{-9} \\ 10^{-9} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1.000001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10^{-10} & 0 \\ 0 & 10^{10} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. Define:

   a. orthogonal matrix

   b. upper Hessenberg matrix

   c. positive definite matrix

   d. $\|x\|_p$, if x is a vector and $1 \leq p < \infty$

   e. $\|A\|_p$, if A is a matrix

6. The following MATLAB program solves a linear system $Ax = b$ with no pivoting, it knocks out all elements above and below the diagonal and then solves the final diagonal system. However, unlike Gauss-Jordan, it knocks out all elements below the diagonal "before" knocking out the elements above. For large N, approximately how many multiplications are done? Show your work.

```
      function X = DLINEQ(A,N,B)
%                                REDUCE TO UPPER TRIANGULAR FORM (NO PIVOTING)
      for I=1:N-1
%                                KNOCK OUT ELEMENTS BELOW DIAGONAL IN COLUMN I
         for J=I+1:N
            LJI = A(J,I)/A(I,I);
            for K=I:N
               A(J,K) = A(J,K) - LJI*A(I,K);
            end
            B(J) = B(J) - LJI*B(I);
         end
      end
%                                NOW REDUCE TO DIAGONAL FORM
      for I=N:-1:2
%                                KNOCK OUT ELEMENTS ABOVE DIAGONAL IN COLUMN I
         for J=1:I-1
            LJI = A(J,I)/A(I,I);
            A(J,I) = A(J,I) - LJI*A(I,I);
            B(J) = B(J) - LJI*B(I);
         end
      end
%                                NOW SOLVE DIAGONAL SYSTEM
      for I=1:N
         X(I) = B(I)/A(I,I);
      end
```