Math 5330, Test I

Name ____________________________

1. If

$$A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 0 & 1 \\ -4 & -4 & 4 \end{bmatrix}$$

do Gaussian elimination *with partial pivoting* to find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $A = PLU$.

2. Prove that the Jacobi method:

$$x_i^{n+1} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^n \right)$$

converges, if A is diagonal dominant ($|a_{ii}| > \sum_{j \neq i} |a_{ij}|$, for each i).

3. What is the order of work ($O(N^\alpha)$) for each of the following? Assume all matrices are N by N, where N is large, and that advantage is taken of any special structure mentioned. Assume A is full, for parts a,b,c,d.

   a. The forward elimination stage of Gaussian elimination applied to $Ax = b$.

   b. The backward substitution stage of Gaussian elimination.

   c. Solution of $Ax = b$ if an LU decomposition is known.

   d. The Gauss-Seidel iteration to solve $Ax = b$, if N iterations are required for convergence.

   e. Solution of $Ax = b$ if A is tridiagonal, except that A_{1N} and A_{N1} are also nonzero.

   f. Solution of $Ax = b$ using Gaussian elimination if A is banded, with bandwidth $\sqrt{N}$, and no pivoting is done.

   g. Same as (f) but now partial pivoting is done.

   h. Same as (f) but now assume an LU decomposition of A is already known.

4. Which of the following linear systems would you expect to produce the most relative round-off error, using Gauss elimination with partial pivoting? Justify your answer.

$$\begin{bmatrix} 10^{-9} & 10^{-8} \\ 10^{-8} & 10^{-9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1.00001 \\ -0.99999 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. A MATLAB program which solves a linear system using Gauss-Jordan does most of its work in the loops:

```
for I=1:N
   for J=1:N
      if (J ~= I)
         for K=I:N
            A(J,K) = A(J,K) - LJI*A(I,K);
         end
      end
   end
end
```

   Approximately how many multiplications are done (show work)? How does this compare to Gaussian elimination?

6. Would you expect the Jacobi iterative method to converge, when used to solve:

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   What about the Gauss-Seidel method? Justify your answers theoretically, that is, without actually taking any iterations.