Math 5330, Test I

Name ______________________________

1. If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 8 \\ -2 & -5 & 4 \end{bmatrix}$$

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $PA = LU$.

2. Use the decomposition of A from Problem 1 to solve $Ax = b$, where $b = (6, 17, -3)$. That is, multiply both sides by $P: \ PAx = Pb$ so $LUx = Pb$, then solve $Ly = Pb$, then $Ux = y$.

3. An N by N band matrix has L non-zero diagonals below the main diagonal and L above. If $1 << L << N$, approximately how many multiplications are done:

   a. during the forward elimination, if no pivoting is done?

   b. during the forward elimination, if partial pivoting is done?

   c. during back substitution, if no pivoting is done?

   d. during back substitution, if partial pivoting is done?

   Hint: below is a MATLAB program which solves a banded linear system with no pivoting. How do the limits change if partial pivoting is done?

```
      function X = LBAND0(A,B,N,L)
%
%     ARGUMENT DESCRIPTIONS
%
%     A - (INPUT) A IS AN N BY 2*L+1 ARRAY CONTAINING THE BAND MATRIX.
%         A(I,L+1+J-I) CONTAINS THE MATRIX ELEMENT IN ROW I, COLUMN J.
%     X - (OUTPUT) X IS THE SOLUTION VECTOR OF LENGTH N.
%     B - (INPUT) B IS THE RIGHT HAND SIDE VECTOR OF LENGTH N.
%     N - (INPUT) N IS THE NUMBER OF EQUATIONS AND NUMBER OF UNKNOWNS
%         IN THE LINEAR SYSTEM.
%     L - (INPUT) L IS THE HALF-BANDWIDTH, DEFINED AS THE MAXIMUM VALUE
%         OF ABS(I-J) SUCH THAT AIJ IS NONZERO.
%
      MD = L+1;
%                               BEGIN FORWARD ELIMINATION
      for K=1:N-1
         if (A(K,MD) == 0.0)
            error ('Zero pivot encountered')
         end
         for I=K+1:min(K+L,N)
            AMUL = -A(I,MD+K-I)/A(K,MD);
            if (AMUL ~= 0.0)
%                               ADD AMUL TIMES ROW K TO ROW I
               for J=K:min(K+L,N)
                  A(I,MD+J-I) = A(I,MD+J-I) + AMUL*A(K,MD+J-K);
               end
               B(I) = B(I) + AMUL*B(K);
            end
         end
```

```
      end
      if (A(N,MD) == 0.0)
         error ('Zero pivot encountered')
      end
%                                 BEGIN BACK SUBSTITUTION
      X(N) = B(N)/A(N,MD);
      for K=N-1:-1:1
         SUM = 0;
         for J=K+1:min(K+L,N)
            SUM = SUM + A(K,MD+J-K)*X(J);
         end
         X(K) = (B(K)-SUM)/A(K,MD);
      end
```

4. Find the condition number of

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1000 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Approximately how many multiplications does the following MATLAB program do, for large N:

```
X = 1;
for L=1:N
   for I=L:N
      for J=L:N
         for K=L:N
            X = X*X;
         end
      end
   end
end
```

6. Consider the 1D boundary value problem $-U_{xx} + U = sin(x)$ with boundary conditions $U(0) = 1, U(2\pi) = 2$. This differential equation can be approximated using the finite difference equation:

$$\frac{-U_{i+1} + 2U_i - U_{i-1}}{h^2} + U_i = sin(x_i)$$

for $i = 2, ..., n$, where $x_i = (i-1)h, h = 2\pi/n$, and U_i is an approximation to $U(x_i)$. The boundary conditions imply $U_1 = 1, U_{n+1} = 2$.

The MATLAB program below should do 1000 iterations of the Gauss-Seidel iteration to solve this finite difference system, complete the incomplete statement. Explain why convergence is guaranteed.

```
      n = 20;
      h = 2*pi/n;
      u(2:n) = 0;
      u(1) = 1;
      u(n+1) = 2;
      for iter=1:1000
         for i=2:n
% complete this statement:
            u(i) =
         end
      end
```

7. Define:

   a. orthogonal matrix

   b. upper Hessenberg matrix

   c. positive definite matrix

   d. $\|A\|_p$, if A is a matrix