Math 5330, Test II

Name ____________________________

1. a. Find a QR decomposition of

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

b. Do one complete iteration of the QR method to the matrix A. Is the new matrix more nearly diagonal, in the sense that the sum of squares of the off-diagonal elements is smaller (note: sum of squares of entire matrix will be the same)?

c. Use the Jacobi method to find all eigenvalues and eigenvectors of A. (Note: only one iteration is necessary!)

2. Prove that if z is a solution to $AA^T z = b$, then $x \equiv A^T z$ is a solution of $Ax = b$ of minimum norm.

3. Under certain conditions, the QR iteration produces a quasitriangular matrix in the limit.

   a. Define a quasitriangular matrix.

   b. In general terms, how do you find the eigenvalues of a quasitriangular matrix?

   c. Is it necessary for convergence, to start the QR iteration from Hessenberg form? What is the advantage of starting from Hessenberg form?

d. If A is symmetric, show that $B = Q^T AQ$ is still symmetric, if Q is an orthogonal matrix. This means that if the original matrix is symmetric, and orthogonal transformations are used to reduce it to upper Hessenberg form, the resulting matrix has what (non-zero) structure? Is $B = M^{-1}AM$ still symmetric, if M is not orthogonal?

e. If A is upper Hessenberg, the work to do one QR iteration is proportional to what power of N (size of matrix)? What if A is tridiagonal and symmetric? What will happen if the QR iteration is applied to a matrix that is tridiagonal and **not** symmetric?

3. Consider the iteration $A_{n+1} = AA_n$, where $A_0 = A$, and assume A is diagonalizable ($A = P^{-1}DP$).

   a. Show that in the limit as $n \to \infty$, $A_{n+1} = \lambda_1 A_n$, where λ_1 is the largest eigenvalue of A in absolute value (assume there is a largest eigenvalue).

   b. The normal power iteration is $v_{n+1} = Av_n$, that is, we normally start with a random vector v_0 and multiply it repeatedly by A, rather than start with the matrix A and multiply it repeatedly by A. What is the advantage of the normal power iteration compared to the alternative approach defined above? Is there any potential disadvantage?

c. Can you think of a way in which the iteration $A_{n+1} = AA_n$ could be made more efficient? (Hint: suppose n is a power of 2)