Math 5330 Final Exam

Name ______________________________

1. If

$$A = \begin{bmatrix} 0 & 3 & 1 \\ -4 & 2 & 1 \\ 8 & 2 & 3 \end{bmatrix}$$

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $A = PLU$.

2. Find a QR decomposition of

$$A = \begin{bmatrix} 12 & -5 \\ -5 & 12 \end{bmatrix}$$

3. Prove the following:

   a. If $A^T A x = A^T b$, then x minimizes $\|Ax - b\|_2$.

b. If $AA^T z = b$, and $x = A^T z$, then x minimizes $\|x\|_2$ over all solutions of $Ax = b$.

c. $I - \frac{2ww^T}{w^T w}$ is orthogonal, for any vector $w \neq 0$.

4. Use the Jacobi method to find all eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

(Note: only one iteration is necessary!)

5. Write the shifted inverse power iteration, used to find the eigenvalue of the generalized problem $Az = \lambda Bz$ closest to a number p, in a form where no inverses appear. How do you calculate the eigenvalue closest to p then?

6. a. Write a linear programming problem which, if solved (but don't solve it), would produce the straight line $y = mx + b$ which most closely fits the data points $(1,1), (2,5), (3,5)$ in the L_∞ norm. Write the constraints in the form $Ax \geq b$. (Hint: you will have 3 unknowns, m, b and ϵ, and 3 constraints involving absolute values, which translate into 6 linear constraints.)

   b. The linear programming problem of part (a) cannot be solved directly by the simplex method because there are not zero bounds on the variables. Write a primal problem, which could be solved by the simplex method, whose dual is the problem in part (a)

7. a. Two factories have 100 and 150 cars, two dealers need 130 and 140 delivered to them. The cost C_{ij} to transport each car from factory i to dealer j is: $C_{11} = 150, C_{12} = 250, C_{21} = 350, C_{22} = 140$. Set up the initial simplex tableaux for the problem of minimizing the cost (but don't solve!), using slack and artificial variables as needed.

   b. Solve this problem. (Hint: if it takes you more than 15 seconds, read the problem again.)

8. Use the simplex method to solve
   max $P = x_1 + x_2 + 2x_3$
   with

$$\begin{aligned} x_1 + 2x_2 - x_3 &\leq 5 \\ 2x_1 + x_2 - x_3 &\leq 8 \end{aligned}$$

   and $x_1, x_2, x_3 \geq 0$

9. What is the order of work for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

   a. One Jacobi iteration (knocking out one subdiagonal element) for finding the eigenvalues of a symmetric matrix A.

   b. The solution of $Ax = b$ using Gaussian elimination, if A is tridiagonal except A_{1N} and A_{N1} are also nonzero.

   c. One QR iteration, if A is upper Hessenberg.

   d. One QR iteration, if A is symmetric and tridiagonal.

   e. One power method iteration.

   f. Normal equations method, to solve $min\|Ax - b\|_2$. Assume A is $2N$ by N.

   g. One simplex step, for solving max $c^T x$ with $Ax \leq b, x \geq 0$.

   h. Solution of $Ax = b$ using Gaussian elimination with partial pivoting, if A is banded, with bandwidth $\sqrt{N}$.

   i. Back substitution, if A is as in part (h).

   j. The orthogonal transformation of a full matrix to a similar upper Hessenberg matrix.

   k. Solution of $Ax = b$ if an LU decomposition of A is known and used.

   l. Solution of $min\|Ax - b\|_2$ if a QR decomposition of A is known and used. Assume A is $2N$ by N.

   m. One Gauss-Seidel iteration, for solving $Ax = b$.

   n. One revised simplex method step, for solving max $c^T x$ with $Ax \leq b, x \geq 0$. Assume A is *not* sparse.

10. *Use the simplex method* with slack and artificial variables as necessary to solve

a. maximize $P = 3x + 4y$
with

$$\begin{aligned} x + y &\leq 3 \\ 2x + y &\leq 4 \end{aligned}$$

and $x, y \geq 0$

b. maximize $P = 3x + 4y$
with

$$\begin{aligned} x + y &\geq 5 \\ 2x + y &\leq 4 \end{aligned}$$

and $x, y \geq 0$