1

Math 5330 Final Exam (a)

Name Key

1. a. Find the straight line $y = mx + b$ which most closely fits the data points $(0,1), (1,5), (2,5)$ in the L_2 norm.

3

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} \approx \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{pmatrix} 11 \\ 15 \end{pmatrix}$$

$$y = 2x + \frac{5}{3} \qquad m = 2, \quad b = \frac{5}{3}$$

b. Write a linear programming problem which, if solved (but don't solve it), would produce the straight line $y = mx + b$ which most closely fits these same data points in the L_∞ norm. Write the constraints in the form $Ax \geq b$. (Hint: you will have 3 unknowns, m, b and ϵ, and 3-constraints involving absolute values, which translate into 6 linear constraints.)

3

minimize ϵ

with $|b + 0m - 1| \leq \epsilon$

$|b + m - 5| \leq \epsilon$

$|b + 2m - 5| \leq \epsilon$

minimize $(0,0,1)^T(b,m,\epsilon)$

$$\begin{pmatrix} -1 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & -2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{bmatrix} b \\ m \\ \epsilon \end{bmatrix} \geq \begin{bmatrix} -1 \\ -5 \\ -5 \\ 1 \\ 5 \\ 5 \end{bmatrix}$$

2. Use the simplex method to solve

a. max $P = 2x_1 + 4x_2 + x_3 + x_4$
with

3

$$\begin{aligned} 2x_1 + x_2 + 2x_3 + 3x_4 &\le 12 \\ 2x_2 + x_3 + 2x_4 &\le 20 \\ 2x_1 + x_2 + 4x_3 &\le 16 \end{aligned}$$

and $x_1, x_2, x_3, x_4 \ge 0$
(Hint: the final basis will consist of x_1, x_2, s_3, where s_3 is the third slack variable; you can use this information to save a lot of work if you want.)

$$\left[\begin{array}{cccccccc|c} 2 & 1 & 2 & 3 & 1 & 0 & 0 & 0 & 12 \\ 0 & 2 & 1 & 2 & 0 & 1 & 0 & 0 & 20 \\ 2 & 1 & 4 & 0 & 0 & 0 & 1 & 0 & 16 \\ \hline -2 & -4 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$ Initial Tableaux

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 16 \end{bmatrix}$$

$\Rightarrow x_1 = 1$
$x_2 = 10$
$s_3 = 4$

$P = 42$

$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0.75 & 1 & 0.5 & -0.25 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 1 & 0 & 0.5 & 0 & 0 & 10 \\ 0 & 0 & 2 & -3 & -1 & 0 & 1 & 0 & 4 \\ \hline 0 & 0 & 2.5 & 5 & 1 & 1.5 & 0 & 1 & 42 \end{array}\right]$$ Final Tableaux

b. max $P = x_1 + x_2 + 2x_3$
with

2

$$\begin{aligned} x_1 + 2x_2 - x_3 &\le 6 \\ 2x_1 + x_2 - x_3 &\le 6 \end{aligned}$$

and $x_1, x_2, x_3 \ge 0$

$$\left[\begin{array}{cccccc|c} 1 & 2 & -1 & 1 & 0 & 0 & 6 \\ 2 & 1 & -1 & 0 & 1 & 0 & 6 \\ -1 & -1 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

unbounded maximum

#3

3. Write the dual problem for problem 2a, and solve it. (Hint: if you use the fact that the dual solution is $y = A_b^{-T} c_b$ you can save yourself a tremendous amount of work.) Use the dual solution to guess what P_{max} for problem 2a would be if the right hand side of the second constraint were increased from 20 to 20.1.

$$\min \; 12y_1 + 20y_2 + 16y_3$$
$$2y_1 \quad + 2y_3 \geq 2$$
$$y_1 + 2y_2 + y_3 \geq 4$$
$$2y_1 + y_2 + 4y_3 \geq 1$$
$$3y_1 + 2y_2 \quad \geq 1$$
$$y_1, y_2, y_3 \geq 0$$

$$y = A_b^{-T} c_b = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}^{-T} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1.5 \\ 0 \end{bmatrix} \qquad y_2 = 1.5$$

$$P_{max} = 42 + 0.1(1.5) = 42.15$$

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4. What is the order of work for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

a. One iteration of the Jacobi method to find the eigenvalues of a symmetric matrix A. $O(N)$

b. Solution of $Ax = b$ using Gaussian elimination, if A is upper Hessenberg. $O(N^2)$

c. One QR iteration, if A is symmetric and tridiagonal. $O(N)$

d. A Fast Fourier Transform, that is, multiplication Ax, where $A_{j,k} = exp(i2\pi(j-1)(k-1)/N)$. $O(N \log_2 N)$

e. A Slow Fourier Transform, that is, multiplication Ax using the usual matrix multiplication formula. $O(N^2)$

f. Solution of min $\|Ax - b\|_2$ using the normal equations, where A is M by N, and $M >> N$. $O(N^2 M)$

g. Solution of $\min \|Ax - b\|_2$ using orthogonal reduction, where A is M by N, and $M >> N$. $O(N^2 M)$

h. One Simplex step, for solving $\max c^T x$ with $Ax \leq b, x \geq 0$, where A is M by N, and $N >> M$. $O(MN)$

i. Solution of $Ax = b$ if an LU decomposition is known. $O(N^2)$

j. One iteration of the inverse power method, for finding the smallest eigenvalue of tridiagonal matrix A. $O(N)$

k. Solution of $Ax = b$ using Gaussian elimination, if A is banded, with bandwidth $N^{\frac{1}{3}}$. $O(N^{5/3})$

5. a. Write a MATLAB (or Fortran) program to efficiently solve $Ax = f$, where A is tridiagonal, and the subdiagonal, diagonal and superdiagonal are stored in vectors a, b, c respectively (see figure below). You may assume pivoting is not necessary.

3

```
function x = trid(a,b,c,N,f)
for i = 1:N-1
   amul = -a(i+1)/b(i);
   b(i+1) = b(i+1) + amul*c(i);
   f(i+1) = f(i+1) + amul*f(i);
end
x(N) = f(N)/b(N);
for i = N-1:-1:1
   x(i) = (f(i) - c(i)*x(i+1))/b(i);
end
```

$$
\begin{bmatrix}
b_1 & c_1 & 0 & 0 & 0 & 0 \\
a_2 & b_2 & c_2 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & a_k & b_k & c_k & 0 & 0 \\
0 & 0 & a_{k+1} & b_{k+1} & c_{k+1} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\
0 & 0 & 0 & 0 & a_N & b_N
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ \cdot \\ x_k \\ x_{k+1} \\ \cdot \\ x_{N-1} \\ x_N
\end{bmatrix}
=
\begin{bmatrix}
f_1 \\ f_2 \\ \cdot \\ f_k \\ f_{k+1} \\ \cdot \\ f_{N-1} \\ f_N
\end{bmatrix}
$$

b. If pivoting is done, describe how your program would change, in general terms (no need to write a new program).

2 Another super-diagonal, d(N), above c, is required for fill-in. Then in the first loop we compare b(i) and a(i+1) and switch rows i and i+1 if |a(i+1)| > |b(i)|.

Math 5330 Final Exam (16)

Name Key

(16)

1. Use the simplex method to solve

a. max $P = 3x_1 + 6x_2 + 10x_3$
with

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\leq 400 \\ 2x_1 + x_2 + 2x_3 &\leq 350 \end{aligned}$$

and $x_1, x_2, x_3 \geq 0$

$$\left[\begin{array}{cccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 400 \\ 2 & 1 & 2 & 0 & 1 & 0 & 350 \\ -3 & -6 & -10 & 0 & 0 & 1 & 0 \end{array}\right]$$

3

$$\to \left[\begin{array}{cccccc|c} 0.5 & 0.75 & 1 & 0.25 & 0 & 0 & 100 \\ 1 & -0.5 & 0 & -0.5 & 1 & 0 & 150 \\ 2 & 1.5 & 0 & 2.5 & 0 & 1 & 1000 \end{array}\right]$$

$x = 0$
$y = 0$
$z = 100$
$\theta = 1000$

b. Find the dual of this problem and set up the *initial* simplex tableaux for this problem, complete with artificial variables. Do not solve. Which is easier to solve, the primal (a) or dual (b)? What is the minimum of the objective for the dual problem? 1000

min $400y_1 + 350y_2$

3

$2y_1 + 2y_2 \geq 3$ $\quad y_i \geq 0$

$3y_2 + y_2 \geq 6$

$4y_1 + 2y_2 \geq 10$

$$\left[\begin{array}{ccccccccc|c} 2 & 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 3 \\ 3 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 4 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 10 \\ 400 & 350 & 0 & 0 & 0 & \alpha & \alpha & \alpha & 1 & 0 \end{array}\right]$$

2. In problem 4.2 you showed that the dual of:

$$\text{maximize } c_1 x_1 + \ldots + c_N x_N$$

with constraints

$$\begin{array}{ccccccc}
a_{1,1}x_1 & + & \ldots & + & a_{1,N}x_N & \leq & b_1, \\
\vdots & & & & \vdots & & \vdots \\
a_{k,1}x_1 & + & \ldots & + & a_{k,N}x_N & \leq & b_k, \\
a_{k+1,1}x_1 & + & \ldots & + & a_{k+1,N}x_N & = & b_{k+1}, \\
\vdots & & & & \vdots & & \vdots \\
a_{M,1}x_1 & + & \ldots & + & a_{M,N}x_N & = & b_M,
\end{array}$$

and bounds

$$\begin{array}{ccc}
x_1 & \geq & 0, \\
\vdots & & \\
x_N & \geq & 0.
\end{array}$$

#3

was

$$\text{minimize } b_1 y_1 + \ldots + b_M y_M$$

with $A^T y \geq c$, and $y_1, \ldots, y_k \geq 0$.

Show directly (without using the fact that these problems are duals) that $b^T y \geq c^T x$ for any dual feasible y and any primal feasible x. If the primal problem has an unbounded maximum, what can we say about the dual problem? (no feasible solution)

equal

$$b^T y = \sum_{i=1}^{K} b_i y_i + \sum_{i=K+1}^{M} b_i y_i \geq \sum_{i=1}^{K} (Ax)_i y_i + \sum_{i=K+1}^{M} (Ax)_i y_i$$

$$= (Ax)^T y = x^T A^T y \geq x^T c$$

3. Consider the points $(0,0), (1,2), (3,1)$.

a. Find the L_2 straight line $y = mx + b$ for these points.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

3

$b = 10/14$

$m = 3/14$

b. Write out a linear programming problem which, if solved, would give the L_1 line for these points. It doesn't need to be in a form that could be solved by the simplex method, and you don't need to solve it. But the constraints should be linear inequalities (or equations), they should not involve absolute values.

min $\epsilon_1 + \epsilon_2 + \epsilon_3$

$|b + 0m - 0| \le \epsilon_1$

$|b + m - 2| \le \epsilon_2$

2 $|b + 3m - 1| \le \epsilon_3$

min $\epsilon_1 + \epsilon_2 + \epsilon_3$

$b - \epsilon_1 \le 0 \qquad b + \epsilon_1 \ge 0$

$b + m - \epsilon_2 \le 2 \qquad b + m + \epsilon_2 \ge 2$

$b + 3m - \epsilon_3 \le 1 \qquad b + 3m + \epsilon_3 \ge 1$

c. Write out a linear programming problem which, if solved, would give the L_∞ line for these points.

min ϵ

$|b + 0m - 0| \le \epsilon$

2 $|b + m - 2| \le \epsilon$

$|b + 3m - 1| \le \epsilon$

min ϵ

$b - \epsilon \le 0 \qquad b + \epsilon \ge 0$

$b + m - \epsilon \le 2 \qquad b + m + \epsilon \ge 2$

$b + 3m - \epsilon \le 1 \qquad b + 3m + \epsilon \ge 1$

4. What is the order of work for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

a. The Jacobi method to find the eigenvalues of a symmetric matrix A. $O(N^3)$

b. Solution of $Ax = b$ using Gaussian elimination, if A is tridiagonal except A_{1N} and A_{N1} are also nonzero. $O(N)$

c. One LR iteration, if A is upper Hessenberg (assume no pivoting). $O(N^2)$

d. One LR iteration, if A is tridiagonal (assume no pivoting). $O(N)$

e. One power method iteration. $O(N^2)$

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f. One inverse power method iteration (not the first iteration), assuming the LU decomposition from the first iteration is saved. $O(N^2)$

g. A Fast Fourier Transform, that is, multiplication Ax, where $A_{j,k} = exp(i2\pi(j-1)(k-1)/N)$. $O(N \log N)$

h. A Slow Fourier Transform, that is, multiplication Ax using the usual matrix multiplication formula. $O(N^2)$

i. One Simplex step, for solving max $c^T x$ with $Ax \leq b, x \geq 0$, where A is M by N, and $N >> M$. $O(MN)$

j. Solution of $Ax = b$ using Gaussian elimination, if A is banded, with bandwidth $N^{\frac{2}{3}}$. $O(N^{7/3})$

k. The orthogonal transformation of a full matrix to a similar upper Hessenberg matrix. $O(N^3)$

5. Explain how you would find the vector x which minimizes $\|Ax - b\|_2$, if you already have the QR decomposition of the M by N matrix A. The operation count would be $O(N^\alpha)$ for what α, if we assume $M \approx 2N$? What would the operation count be if you don't have a QR decomposition? $\alpha = 2$

#3

$\alpha = 3$

$QRx \cong b$

$Rx \cong Q^T b$ $O(N^2)$ work

solve upper triangular system, also $O(N^2)$ work

Math 5330 Final Exam (c)

Name Key

1. a. A bicycle manufacturer builds one-, three- and ten-speed models. The models need 20,30 and 40 units of steel, respectively, and 12,21 and 16 units of aluminum, and the company has available 91,800 units of steel and 42,000 units of aluminum. How many of each model should be made to maximize the profit, if the company makes $8 per one-speed, $12 per three-speed, and $24 per ten-speed? What is the maximum possible profit?

max $8a + 12b + 24c = P$

$20a + 30b + 40c \le 91800$ $\quad a, b, c \ge 0$

$12a + 21b + 16c \le 42000$

3

$$\left[\begin{array}{cccccc|c} 20 & 30 & 40 & 1 & 0 & 0 & 91800 \\ 12 & 21 & 16 & 0 & 1 & 0 & 42000 \\ -8 & -12 & -24 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccccc|c} 0.5 & 0.75 & 1 & 0.025 & 0 & 0 & 2295 \\ 4 & 9 & 0 & -0.4 & 1 & 0 & 5280 \\ 4 & 6 & 0 & 0.6 & 0 & 1 & 55080 \end{array}\right]$$

$a = 0$

$b = 0$ $\quad P = 55080$

$c = 2295$

note:

$$\begin{bmatrix} c \\ s_2 \end{bmatrix} = \begin{bmatrix} 40 & 0 \\ 16 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 91800 \\ 42000 \end{bmatrix} = \begin{bmatrix} 2295 \\ 5280 \end{bmatrix}$$

Note

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{pmatrix} 40 & 0 \\ 16 & 1 \end{pmatrix}^{-T} \begin{bmatrix} 24 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}$$

2

b. Write the (symmetric) dual for the above primal problem. Would the dual be easier to solve than the primal using the simplex method? Explain. min $91800\,y_1 + 42000\,y_2$

no, artificial variable needed

$$20y_1 + 12y_2 \geq 8 \qquad y_1 \geq 0$$
$$30y_1 + 21y_2 \geq 12 \qquad y_2 \geq 0$$
$$40y_1 + 16y_2 \geq 24$$

c. The dual solution is y = (0.6,0.0). Explain the significance of the two components of y, for this application.

1

Profit will increase 0.6 for each extra unit of steel
0 " alum.

2. Two factories have 300 and 400 cars, three dealers need 100, 120 and 60 delivered to them. The cost C_{ij} to transport each car from factory i to dealer j is: $C_{11} = 100, C_{12} = 150, C_{13} = 180, C_{21} = 250, C_{22} = 240, C_{23} = 180$. Set up the initial simplex tableaux for this problem (but don't solve!), using slack and artificial variables as needed.

No

2

$$\begin{array}{cccccccccccc|c}
x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} & s_1 & s_2 & a_1 & a_2 & a_3 & P & \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 300 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 400 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 100 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 120 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 60 \\
\hline
100 & 150 & 180 & 250 & 240 & 180 & 0 & 0 & \alpha & \alpha & \alpha & 1 & 0
\end{array}$$

3. What is the order of work for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

   $O(N)$ a. One Jacobi iteration for finding the eigenvalues of a symmetric matrix A.

   $O(N)$ b. The solution of $Ax = b$ using Gaussian elimination, if A is tridiagonal except A_{1N} and A_{N1} are also nonzero.

   $O(N^2)$ c. One QR iteration, if A is upper Hessenberg.

   $O(N)$ d. One QR iteration, if A is symmetric and tridiagonal.

   $O(N^2)$ e. One power method iteration.

   $O(N^3)$ f. Normal equations method, to solve $min\|Ax - b\|_2$. Assume A is $2N$ by N.

   $O(N \log N)$ g. A fast Fourier transform, that is, multiplication Ax, where $A_{j,k} = exp(i2\pi(j-1)(k-1)/N)$, and N is a power of 2.

6

   $O(N^2)$ h. A slow Fourier transform, that is, multiplication Ax using the usual matrix multiplication formula.

   $O(N^2)$ i. One simplex step, for solving max c^Tx with $Ax \leq b, x \geq 0$.

   $O(N^2)$ j. Solution of $Ax = b$ using Gaussian elimination with partial pivoting, if A is banded, with bandwidth $\sqrt{N}$.

   $O(N^{3/2})$ k. Back substitution, if A is as in part (j).

   $O(N^3)$ l. The orthogonal transformation of a full matrix to a similar upper Hessenberg matrix.

   $O(N^2)$ m. Solution of $Ax = b$ if an LU decomposition of A is known and used.

   $O(N^2)$ n. Solution of $min\|Ax - b\|_2$ if a QR decomposition of A is known and used. Assume A is $2N$ by N.

   $O(N^2)$ o. One Gauss-Seidel iteration, for solving $Ax = b$.

4. True or False

   False a. It is possible to reduce a general 5 by 5 symmetric matrix to diagonal form in a finite number of steps, using the Jacobi (eigenvalue) algorithm.

False b. If λ is the largest eigenvalue of $(A - pI)^{-1}$, then $p - \frac{1}{\lambda}$ is the eigenvalue of A closest to p.

True c. If A is symmetric and positive definite, its singular values are the same as its eigenvalues.

False d. The simplex method is guaranteed to converge in at most $3N$ steps, where N=number of unknowns, assuming there is a solution.

True e. If A is diagonal dominant, both the Jacobi and Gauss-Seidel iterations for solving $Ax = b$ are guaranteed to converge, for any starting guess.

False f. The revised simplex method is generally faster when the constraint matrix is sparse and has many more equations than unknowns.

True g. The fast Fourier transform is more efficient when N is a power of 2, than when N is prime.

False h. If $x = A^T(AA^T)^{-1}b$, then x is the solution to min $\|Ax - b\|_2$, assuming the indicated inverse exists.

6 False i. If $x = (A^TA)^{-1}A^Tb$, then x is the solution to $Ax = b$ of minimum 2-norm, assuming the indicated inverse exists.

False j. If A has a condition number of 10, we should expect to lose about 10 significant decimal digits in solving $Ax = b$ with Gaussian elimination and partial pivoting.

True k. The SOR method is guaranteed to converge if A is symmetric and positive definite, and $0 < \omega < 2$.

True l. The generalized eigenvalue problem $Ax = \lambda Bx$ may have complex eigenvalues, even if A and B are symmetric.

True m. If all eigenvalues have distinct absolute values, the QR iteration will converge to triangular form, even if you start from a full matrix.

True n. The revised simplex method is more efficient than the usual simplex method, for transportation problems.

True o. A fast Fourier tranform routine can be used to find an inverse Fourier transform, with little additional effort.

Math 5330 Test III (2)

Name Key

Solve any 5 of the 6 problems.

1. Write a linear programming problem which, if solved (but don't solve it), would produce the straight line $y = mx + b$ which most closely fits the data points $(0,1), (1,5), (2,5)$ in the L_∞ norm. Write the constraints in the form $Ax \geq b$. (Hint: you will have 3 unknowns, m, b and ϵ, and 3 constraints involving absolute values, which translate into 6 linear constraints.)

min ϵ

with $|b + 0m - 1| \leq \epsilon$

$|b + m - 5| \leq \epsilon$

$|b + 2m - 5| \leq \epsilon$

min $(0, 0, 1)^T (b, m, \epsilon)$

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & -2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} b \\ m \\ \epsilon \end{bmatrix} \geq \begin{bmatrix} -1 \\ -5 \\ -5 \\ 1 \\ 5 \\ 5 \end{bmatrix}$$

2. Two factories have 300 and 400 cars, three dealers need 100, 120 and 60 delivered to them. The cost C_{ij} to transport each car from factory i to dealer j is: $C_{11} = 100, C_{12} = 150, C_{13} = 180, C_{21} = 250, C_{22} = 240, C_{23} = 180$. Set up the initial simplex tableaux for this problem (but don't solve!), using slack and artificial variables as needed.

$$\begin{array}{cccccccccccc|c}
x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} & s_1 & s_2 & a_1 & a_2 & a_3 & P & \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 300 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 400 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 100 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 120 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 60 \\
100 & 150 & 180 & 250 & 240 & 180 & 0 & 0 & \alpha & \alpha & \alpha & 1 & 0
\end{array}$$

3. Use the simplex method to solve
max $P = 2x_1 + 4x_2 + x_3 + x_4$
with

$$\begin{aligned} 2x_1 + x_2 + 2x_3 + 3x_4 &\leq 12 \\ 2x_2 + x_3 + 2x_4 &\leq 20 \\ 2x_1 + x_2 + 4x_3 \quad &\leq 16 \end{aligned}$$

and $x_1, x_2, x_3, x_4 \geq 0$
(Hint: the final basis will consist of x_1, x_2, s_3, where s_3 is the third slack variable; you can use this information to save a lot of work if you want.)

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 16 \end{bmatrix}$$

$x_1 = 1$
$x_2 = 10$
$x_3 = 0$
$x_4 = 0$
$s_3 = 4$

$(P = 42)$

4. Use the simplex method to solve
max $P = x_1 + x_2 + 2x_3$
with

$$\begin{aligned} x_1 + 2x_2 - x_3 &\leq 6 \\ 2x_1 + x_2 - x_3 &\leq 6 \end{aligned}$$

and $x_1, x_2, x_3 \geq 0$

$$\left[\begin{array}{cccccc|c} 1 & 2 & -1 & 1 & 0 & 0 & 6 \\ 2 & 1 & -1 & 0 & 1 & 0 & 6 \\ -1 & -1 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$\uparrow_2$

unbounded max

5. Use the simplex method to solve
max $P = 3x_1 + 6x_2 + 10x_3$
with

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\leq 400 \\ 2x_1 + x_2 + 2x_3 &\leq 350 \end{aligned}$$

and $x_1, x_2, x_3 \geq 0$

$$\left[\begin{array}{cccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 400 \\ 2 & 1 & 2 & 0 & 1 & 0 & 350 \\ -3 & -6 & -10 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccccc|c} 0.5 & 0,75 & 1 & 0,25 & 0 & 0 & 100 \\ 1 & -0,5 & 0 & -0,5 & 1 & 0 & 150 \\ 2 & 1,5 & 0 & 2,5 & 0 & 1 & 1000 \end{array}\right]$$

$x = 0$
$y = 0$
$z = 100$

$P = 1000$

6. a. Find the (symmetric) dual of problem 5 and set up the *initial* simplex tableaux for this problem, complete with artificial variables. Do not solve. Which is easier to solve, the primal (a) or dual (b)? What is the minimum of the objective for the dual problem?

$$\left[\begin{array}{ccccccccc|c} 2 & 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 3 \\ 3 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 4 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 10 \\ 400 & 350 & 0 & 0 & 0 & \alpha & \alpha & \alpha & 1 & 0 \end{array}\right]$$

b. The solution of the dual problem is $y = (2.5, 0)$. Knowing this, if the 400 on the right hand side of the first inequality in (a) were increased to 401 and the problem were re-solved, what what the new P be?

$P = 1000 + 2,5 = 1002,5$

Math 5330 Final Exam (e)

Name Key

12

1. What is the order of work for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

   a. One iteration (knock out one element) of the Jacobi method to find the eigenvalues of a symmetric matrix A. $O(N)$

   b. Solution of $Ax = b$ using Gaussian elimination, if A is upper Hessenberg. $O(N^2)$

   c. One QR iteration, if A is full. $O(N^3)$

   d. One LR iteration, if A is upper Hessenberg. $O(N^2)$

   e. One QR iteration, if A is symmetric and tridiagonal. $O(N)$

   f. Reduction to upper Hessenberg form, using orthogonal similarity transformations. $O(N^3)$

   g. Solution of min $\|Ax - b\|_2$ using the normal equations, where A is M by N, and $M >> N$. $O(N^2M)$

   h. Solution of min $\|Ax - b\|_2$ using orthogonal reduction, where A is M by N, and $M >> N$. $O(N^2M)$

   i. One simplex step, for solving max c^Tx with $Ax \leq b, x \geq 0$, where A is M by N, and $N >> M$. $O(NM)$

   j. Solution of $Ax = b$ if an LU decomposition is known. $O(N^2)$

   k. One iteration of the inverse power method, for finding the smallest eigenvalue of tridiagonal matrix A. $O(N)$

   l. Solution of $Ax = b$ using Gaussian elimination, if A is banded, with bandwidth $N^{\frac{1}{3}}$. $(O(N^{5/3}))$

2. Use the simplex method to solve:

max $P = 3x + 4y$
with

$$\begin{aligned} x + y &\leq 6 \\ 2x + y &\leq 8 \end{aligned}$$

and $x, y \geq 0$

5

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 6 \\ 2 & 1 & 0 & 1 & 0 & 8 \\ -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \begin{array}{c} 0\ 6\ 0\ 2\ 24 \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 6 \\ 1 & 0 & -1 & 1 & 0 & 2 \\ 1 & 0 & 4 & 0 & 1 & 24 \end{array}\right] \end{array}$$

$x = 0$
$y = 6$ $\quad P = 24$

3. Write the (symmetric) dual to the previous problem, and set up the initial simplex tableaux, with slack and artificial variables.

min $D = 6x + 8y + \alpha a_1 + \alpha a_2$

$x + 2y \geq 3$

$x + y \geq 4$

$x \geq 0$

$y \geq 0$

4

$$\begin{array}{ccccccc|c} x & y & s_1 & s_2 & a_1 & a_2 & P & \\ \end{array}$$
$$\left[\begin{array}{ccccccc|c} 1 & 2 & -1 & 0 & 1 & 0 & 0 & 3 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 & 4 \\ +6 & +8 & 0 & 0 & +\alpha & +\alpha & 1 & 0 \end{array}\right]$$

max $-6x - 8y - \alpha a_1 - \alpha a_2$

4. Find the straight line $y = p + qx$ which most closely fits the data points $(0,1), (1,6), (2,2)$ in the L_2 norm.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$$

4

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix} \qquad p = \frac{5}{2}, \quad q = \frac{1}{2}$$

5. Find A, b, c such that the following LP problem, if solved, would produce the straight line which most closely fits the data points of problem 4 in the L_1 norm.

minimize $b^T y$, with $A^T y \geq c$.

Here $y = [p, q, \epsilon_1, \epsilon_2, \epsilon_3]$ is the vector of unknowns. (Note: the dual of this problem would be: maximize $c^T x$, with $Ax \leq b, x \geq 0$, which could actually be solved by the simplex method.)

5

$$\begin{aligned} &\min\ \epsilon_1 + \epsilon_2 + \epsilon_3 \\ &p + 0q - 1 \leq \epsilon_1 \\ &p + 0q - 1 \geq -\epsilon_1 \\ &p + 1q - 6 \leq \epsilon_2 \\ &p + 1q - 6 \geq -\epsilon_2 \\ &p + 2q - 2 \leq \epsilon_3 \\ &p + 2q - 2 \geq -\epsilon_3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\min\ \epsilon_1 + \epsilon_2 + \epsilon_3 \\ &-p + 0q + \epsilon_1 \geq -1 \\ &-p - q + \epsilon_2 \geq -6 \\ &-p - 2q + \epsilon_3 \geq -2 \\ &p + 0q + \epsilon_1 \geq 1 \\ &p + q + \epsilon_2 \geq 6 \\ &p + 2q + \epsilon_3 \geq 2 \end{aligned}$$

$$A = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad b = [0, 0, 1, 1, 1] \qquad c = [-1, -6, -2, 1, 6, 2]$$

Math 5330 Final (f)

Name Key

1. Use the simplex method to solve

a. max $P = 3x_1 + 6x_2 + 10x_3$
with

$$2x_1 + 3x_2 + 4x_3 \leq 400$$
$$2x_1 + x_2 + 2x_3 \leq 350$$

and $x_1, x_2, x_3 \geq 0$

$$\left[\begin{array}{cccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 400 \\ 2 & 1 & 2 & 0 & 1 & 0 & 350 \\ \hline -3 & -6 & -10 & 0 & 0 & 1 & 0 \end{array}\right]$$

4

$$\rightarrow \left[\begin{array}{cccccc|c} 0.5 & 0.75 & 1 & 0.25 & 0 & 0 & 100 \\ 1 & -0.5 & 0 & -0.5 & 1 & 0 & 150 \\ \hline 2 & 1.5 & 0 & 2.5 & 0 & 1 & 1000 \end{array}\right]$$

$x = 0$
$y = 0$ $\quad P = 1000$
$z = 100$

b. Find the (symmetric) dual of this problem and solve it graphically.

min $C = 400y_1 + 350y_2$

$2y_1 + 2y_2 \geq 3$

$3y_1 + y_2 \geq 6$

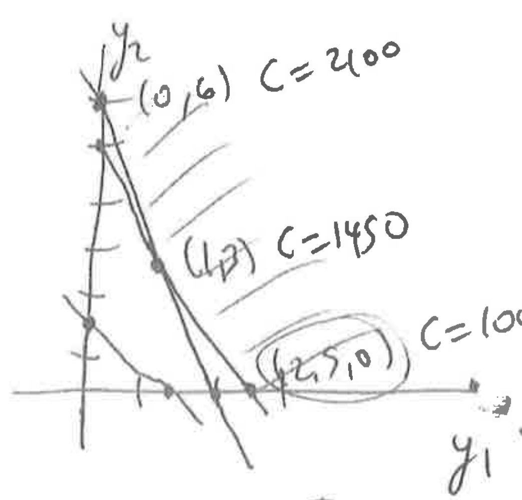
$4y_1 + 2y_2 \geq 10$

$y_1 \geq 0$

$y_2 \geq 0$

4

$y_1 = 2.5$
$y_2 = 0$



(or) $y = A_b^{-T} c_b = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}^{-T} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$

c. What is the significance of y_1 (the first component of the dual solution), relative to the original problem?

2 y_1 = increase in P per unit b_1 increased (400 +)

2. Given that the primal problem:

$$\text{maximize } P = c^T x$$

with constraints $Ax = b$ and bounds $x \geq 0$ has dual:

$$\text{minimize } D = b^T y$$

with constraints $A^T y \geq c$

a. What is the dual of the primal problem:

$$\text{maximize } P = c^T x$$

with constraints

$$\begin{array}{ccccccc} a_{1,1}x_1 & + & \dots & + & a_{1,N}x_N & \leq & b_1, \\ \vdots & & & & \vdots & & \vdots \\ a_{k,1}x_1 & + & \dots & + & a_{k,N}x_N & \leq & b_k, \\ a_{k+1,1}x_1 & + & \dots & + & a_{k+1,N}x_N & = & b_{k+1}, \\ \vdots & & & & \vdots & & \vdots \\ a_{M,1}x_1 & + & \dots & + & a_{M,N}x_N & = & b_M, \end{array}$$

4

and bounds $x \geq 0$?

min $b^T y$ with $A^T y \geq c$ $y_1 \cdots y_n \geq 0$

$$\begin{bmatrix} a_{11} & \cdots & a_{k1} & \cdots & a_{M1} \\ \vdots & & \vdots & & \vdots \\ a_{1N} & & a_{kN} & & a_{MN} \\ 1 & & 0 & & 0 \\ 0 & & \vdots & & \vdots \\ \vdots & & 0 & & \\ 0 & & 1 & & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \geq \begin{bmatrix} c_1 \\ \vdots \\ c_N \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

b. Show directly (without using the fact that the problems are duals) that the minimum of the dual problem is greater than or equal to the maximum of the primal problem (they are actually equal, but you don't need to show that). If the primal problem is unbounded (maximum is infinite) what can we say about the dual problem?

If x is primal feasible and y is dual feasible:

$$\sum_{i=1}^{K} b_i y_i \geq \sum_{i=1}^{K} (Ax)_i y_i \qquad \text{since } \begin{cases} (Ax)_i \leq b_i \\ y_i \geq 0 \end{cases}$$

$$\sum_{i=K+1}^{M} b_i y_i = \sum_{i=K+1}^{M} (Ax)_i y_i \qquad \text{since } (Ax)_i = b_i$$

$$\Rightarrow \; b^T y \geq (Ax)^T y = x^T A^T y \geq x^T c \qquad \text{since } \begin{cases} A^T y \geq c \\ x \geq 0 \end{cases}$$

If primal is unbounded, dual is infeasible

3. Prove that the Gauss–Seidel iteration:
$x_i^{n+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{n+1} - \sum_{j>i} a_{ij}x_j^{n}\right)$
converges when A is diagonal-dominant, that is, when $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$, for each i. Note that the exact solution satisfies:
$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j - \sum_{j>i} a_{ij}x_j\right)$

$$r_i + s_i < 1 \Rightarrow s_i < 1 - r_i$$

Define $e_i^n = x_i^n - x_i \qquad r_i = \sum_{j<i} \left|\frac{a_{ij}}{a_{ii}}\right| \qquad s_i = \sum_{j>i} \left|\frac{a_{ij}}{a_{ii}}\right|$

$$e_i^{n+1} = -\sum_{j<i} \frac{a_{ij}}{a_{ii}} e_j^{n+1} - \sum_{j>i} \frac{a_{ij}}{a_{ii}} e_j^n$$

$$|e_i^{n+1}| \leq \|e^{n+1}\|_\infty r_i + \|e^n\|_\infty s_i$$

for some i $\rightarrow$

$$\|e^{n+1}\|_\infty = |e_i^{n+1}| \leq \|e^{n+1}\|_\infty r_i + \|e^n\|_\infty s_i$$

$$\|e^{n+1}\|_\infty \leq \frac{s_i}{1-r_i} \|e^n\|_\infty < \|e^n\|_\infty$$

4. a. Write a linear programming problem which, if solved (but don't solve it), would produce the straight line $y = mx + b$ which most closely fits the data points $(0,1), (1,5), (2,5)$ in the L_1 norm. Write the constraints in the form $Ax \geq b$. (Hint: you will have 5 unknowns, $m, b, \epsilon_1, \epsilon_2, \epsilon_3$, and 3 constraints involving absolute values, which should be translated into 6 linear constraints.)

min $\epsilon_1 + \epsilon_2 + \epsilon_3$

with
$$|b + 0m - 1| \leq \epsilon_1$$
$$|b + m - 5| \leq \epsilon_2$$
$$|b + 2m - 5| \leq \epsilon_3$$

or

min $\epsilon_1 + \epsilon_2 + \epsilon_3$
$$-b - 0m + \epsilon_1 \geq -1$$
$$b + 0m + \epsilon_1 \geq 1$$
$$-b - m + \epsilon_2 \geq -5$$
$$b + m + \epsilon_2 \geq 5$$
$$-b - 2m + \epsilon_3 \geq -5$$
$$b + 2m + \epsilon_3 \geq 5$$

4

b. Find the line $y = mx + b$ which most closely fits these same data point in the L_2 norm.

4

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

$$\begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{bmatrix} 15 \\ 11 \end{bmatrix}$$

$m = 2$
$b = \frac{5}{3}$

Final

Math 5330 Exam (g)

Name Key

1. If

$$A = \begin{bmatrix} 0 & 3 & 1 \\ -4 & 2 & 1 \\ 8 & 2 & 3 \end{bmatrix}$$

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $A = PLU$.

2

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} -4 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

or $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 8 & 2 & 3 \\ 0 & 3 & \frac{5}{2} \\ 0 & 0 & -\frac{3}{2} \end{bmatrix}$$

2. Find a QR decomposition of

$$A = \begin{bmatrix} 12 & -5 \\ -5 & 12 \end{bmatrix}$$

2

$$\begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{bmatrix}\begin{bmatrix} 13 & -\frac{120}{13} \\ 0 & \frac{119}{13} \end{bmatrix}$$

3. Prove the following:

a. If $A^T Ax = A^T b$, then x minimizes $\|Ax - b\|_2$.

2

$$\|A(x+e) - b\|^2 = (Ax - b + Ae)^T (Ax - b + Ae)$$

$$= \|Ax - b\|^2 + 2e^T \underbrace{\cancel{A^T(Ax - b)}}_{0} + \|Ae\|^2$$

$$\geq \|Ax - b\|^2$$

b. If $AA^Tz = b$, and $x = A^Tz$, then x minimizes $\|x\|_2$ over all solutions of $Ax = b$. $Ax = b$ obviously. Suppose $Ay = b$ also, $e \equiv y - x$

$$\|y\|^2 = (x+e)^T(x+e) = \|x\|^2 + 2x^Te + \|e\|^2$$

2 $$= \|x\|^2 + 2z^TAe + \|e\|^2 = \|x\|^2 + 2z^T(\overset{\to 0}{Ay - Ax}) + \|e\|^2$$

$$= \|x\|^2 + \|e\|^2 \geq \|x\|^2$$

c. $I - \frac{2ww^T}{w^Tw}$ is orthogonal, for any vector $w \neq 0$.

$$\left(I - \frac{2ww^T}{w^Tw}\right)^T\left(I - \frac{2ww^T}{w^Tw}\right) = I - 4\frac{ww^T}{w^Tw} + \frac{4w(w^Tw)w^T}{(w^Tw)^2}$$

2 $$= I - 4\frac{ww^T}{w^Tw} + 4\frac{ww^T}{w^Tw} = I$$

4. Use the Jacobi method to find all eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

(Note: only one iteration is necessary!)

2 $$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} 4-6cs & 3(c^2-s^2) \\ 3(c^2-s^2) & 4+6cs \end{bmatrix}$$

$c = s = \frac{1}{\sqrt{2}}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$$

$\lambda = 1$ $z = \begin{pmatrix} c \\ -s \end{pmatrix} \in \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 7$ $z = \begin{pmatrix} s \\ c \end{pmatrix} \in \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5. Write the shifted inverse power iteration, used to find the eigenvalue of the generalized problem $Az = \lambda Bz$ closest to a number p, in a form where no inverses appear. How do you calculate the eigenvalue closest to p then? $B^{-1}Az = \lambda z$ $z_{n+1} = (B^{-1}A - pI)^{-1} z_n$

2

$(B^{-1}A - pI) z_{n+1} = z_n$ $(A - pB) z_{n+1} = B z_n$

$\lambda_p = p + \frac{1}{\mu_{max}}$ μ_{max} = largest eigen of $(B^{-1}A - pI)^{-1}$

6. a. Write a linear programming problem which, if solved (but don't solve it), would produce the straight line $y = mx + b$ which most closely fits the data points $(1,1), (2,5), (3,5)$ in the L_∞ norm. Write the constraints in the form $Ax \geq b$. (Hint: you will have 3 unknowns, m, b and ϵ, and 3 constraints involving absolute values, which translate into 6 linear constraints.)

min ϵ min $0m + 0b + \epsilon$

2

$|m + b - 1| \leq \epsilon$

$|2m + b - 5| \leq \epsilon$

$|3m + b - 5| \leq \epsilon$

$$\begin{pmatrix} -1 & -1 & 1 \\ -2 & -1 & 1 \\ -3 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \\ \epsilon \end{pmatrix} \geq \begin{pmatrix} -1 \\ -5 \\ -5 \\ 1 \\ 5 \\ 5 \end{pmatrix}$$

b. The linear programming problem of part (a) cannot be solved directly by the simplex method because there are not zero bounds on the variables. Write a primal problem, which could be solved by the simplex method, whose dual is the problem in part (a)

max $(-1, -5, -5, 1, 5, 5) \cdot (x_1, x_2, x_3, x_4, x_5, x_6)$

2

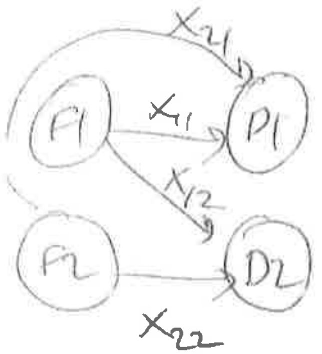
$$\begin{pmatrix} -1 & -2 & -3 & 1 & 2 & 3 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$x_i \geq 0$

max $P = -150x_{11} - 250x_{12} - 350x_{21} - 140x_{22} - \alpha a_1 - \alpha a_2$

2

7. a. Two factories have 100 and 150 cars, two dealers need 130 and 140 delivered to them. The cost C_{ij} to transport each car from factory i to dealer j is: $C_{11} = 150, C_{12} = 250, C_{21} = 350, C_{22} = 140$. Set up the initial simplex tableaux for the problem of minimizing the cost (but don't solve!), using slack and artificial variables as needed.



$$
\begin{array}{ccccccccc|c}
x_{11} & x_{12} & x_{21} & x_{22} & s_1 & s_2 & a_1 & a_2 & P & \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 100 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 150 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 130 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 140 \\
\hline
150 & 250 & 350 & 140 & 0 & 0 & \alpha & \alpha & 1 & 0
\end{array}
$$

$x_{11} + x_{12} \le 100$

$x_{21} + x_{22} \le 150$

$x_{11} + x_{21} = 130$

$x_{12} + x_{22} = 140$

1

b. Solve this problem. (Hint: if it takes you more than 15 seconds, read the problem again.)

no feasible solution

8. Use the simplex method to solve
max $P = x_1 + x_2 + 2x_3$
with

2

$$
\begin{aligned}
x_1 + 2x_2 - x_3 &\le 5 \\
2x_1 + x_2 - x_3 &\le 8
\end{aligned}
$$

and $x_1, x_2, x_3 \ge 0$

$$
\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & P & \\
\hline
1 & 2 & -1 & 1 & 0 & 0 & 5 \\
2 & 1 & -1 & 0 & 1 & 0 & 8 \\
\hline
-1 & -1 & -2 & 0 & 0 & 1 & 0 \\
 & & \uparrow & & & &
\end{array}
$$

unbounded maximum

9. What is the order of work for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

7

- N a. One Jacobi iteration (knocking out one subdiagonal element) for finding the eigenvalues of a symmetric matrix A.
- N b. The solution of $Ax = b$ using Gaussian elimination, if A is tridiagonal except A_{1N} and A_{N1} are also nonzero.
- N^2 c. One QR iteration, if A is upper Hessenberg.
- N d. One QR iteration, if A is symmetric and tridiagonal.
- N^2 e. One power method iteration.
- N^3 f. Normal equations method, to solve $min\|Ax - b\|_2$. Assume A is $2N$ by N.
- N^2 g. One simplex step, for solving max $c^T x$ with $Ax \leq b, x \geq 0$.
- N^2 h. Solution of $Ax = b$ using Gaussian elimination with partial pivoting, if A is banded, with bandwidth $\sqrt{N}$.
- $N^{1.5}$ i. Back substitution, if A is as in part (h).
- N^3 j. The orthogonal transformation of a full matrix to a similar upper Hessenberg matrix.
- N^2 k. Solution of $Ax = b$ if an LU decomposition of A is known and used.
- N^2 l. Solution of $min\|Ax - b\|_2$ if a QR decomposition of A is known and used. Assume A is $2N$ by N.
- N^2 m. One Gauss-Seidel iteration, for solving $Ax = b$.
- N^2 n. One revised simplex method step, for solving max $c^T x$ with $Ax \leq b, x \geq 0$. Assume A is *not* sparse.

10. *Use the simplex method* with slack and artificial variables as necessary to solve

a. maximize $P = 3x + 4y$
with

$$\begin{aligned} x + y &\le 3 \\ 2x + y &\le 4 \end{aligned}$$

and $x, y \ge 0$

$$\begin{array}{ccccc} x & y & s_1 & s_2 & P \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 1 & 0 & 4 \\ \hline -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

3

$$\to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ \hline 1 & 0 & 4 & 0 & 1 & 12 \end{array}\right]$$

$x = 0$
$y = 3$ $P = 12$

b. maximize $P = 3x + 4y$
with

$$\begin{aligned} x + y &\ge 5 \\ 2x + y &\le 4 \end{aligned}$$

and $x, y \ge 0$

3

$$\begin{array}{cccccc} x & y & s_1 & s_2 & a_1 & P \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 1 & -1 & 0 & 1 & 0 & 5 \\ 2 & 1 & 0 & 1 & 0 & 0 & 4 \\ \hline -3 & -4 & 0 & 0 & \alpha & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccccc|c} 1 & 1 & -1 & 0 & 1 & 0 & 5 \\ 2 & 1 & 0 & 1 & 0 & 0 & 4 \\ \hline -3-\alpha & -4-\alpha & \alpha & 0 & 0 & 1 & -5\alpha \end{array}\right] \to \left[\begin{array}{cccccc|c} -1 & 0 & -1 & -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 & 0 & 4 \\ \hline 5+\alpha & 0 & \alpha & 4+\alpha & 0 & 1 & 16-\alpha \end{array}\right]$$

no feasible solution

Math 5330 Final Exam (i)

Name Key

1. a. Find the straight line $y = mx + b$ which most closely fits the data points $(0,1), (1,5), (2,5)$ in the L_2 norm.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} \approx \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \end{bmatrix}$$

$m = 2$

$b = 5/3$

$y = 2x + \frac{5}{3}$

5

b. Write a linear programming problem which, if solved (but don't solve it), would produce the straight line $y = mx + b$ which most closely fits these same data points in the L_1 norm. Write the constraints in the form $Ax \geq b$. (Hint: you will have 5 unknowns, $m, b, \epsilon_1, \epsilon_2, \epsilon_3$, and 3 constraints involving absolute values, which translate into 6 linear constraints.)

10

$\min \ \epsilon_1 + \epsilon_2 + \epsilon_3$

$|b + 0m - 1| \leq \epsilon_1$

$|b + 1m - 5| \leq \epsilon_2$

$|b + 2m - 5| \leq \epsilon_3$

$\min \ (0,0,1,1,1)^T (b, m, \epsilon_1, \epsilon_2, \epsilon_3)$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ m \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 5 \\ 5 \\ -1 \\ -5 \\ -5 \end{pmatrix}$$

2. Use the simplex method to solve

a. max $P = 2x_1 + 4x_2 + x_3 + x_4$
with

$$\begin{aligned} 2x_1 + x_2 + 2x_3 + 3x_4 &\leq 12 \\ 2x_2 + x_3 + 2x_4 &\leq 20 \\ 2x_1 + x_2 + 4x_3 &\leq 16 \end{aligned}$$

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and $x_1, x_2, x_3, x_4 \geq 0$
(Hint: the final basis will consist of x_1, x_2, s_3, where s_3 is the third slack variable; you can use this information to save a lot of work if you want.)

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 16 \end{bmatrix} \rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 10 \\ s_3 = 4 \end{matrix} \quad P = 42$$

b. max $P = x_1 + 3x_2 + 2x_3$
with

$$\begin{aligned} x_1 - 2x_2 + x_3 &\leq 2 \\ 2x_1 \qquad + x_3 &\leq 6 \end{aligned}$$

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and $x_1, x_2, x_3 \geq 0$

$$\left[\begin{array}{cccccc|c} 1 & -2 & 1 & 1 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 & 1 & 0 & 6 \\ \hline -1 & -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$\uparrow$

unbounded max

c. What is the dual problem of [b.] and what is its solution?

$\min\ 2y_1 + 6y_2$

S

$$\begin{aligned} y_1 + 2y_2 &\geq 1 \\ -2y_1 &\geq 3 \qquad y_1, y_2 \geq 0 \\ y_1 + y_2 &\geq 2 \end{aligned}$$

no feasible solution

3. Two factories have 300 and 400 cars, three dealers need 100, 120 and 60 delivered to them. The cost C_{ij} to transport each car from factory i to dealer j is: $C_{11} = 100, C_{12} = 150, C_{13} = 180, C_{21} = 250, C_{22} = 240, C_{23} = 180$. Set up the initial simplex tableaux for this problem (but don't solve!), using slack and artificial variables as needed.

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$$\begin{array}{cccccccccccc|c}
x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} & s_1 & s_2 & a_1 & a_2 & a_3 & P & \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 300 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 400 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 100 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 120 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 60 \\
\hline
100 & 150 & 180 & 250 & 240 & 180 & 0 & 0 & \alpha & \alpha & \alpha & 1 & 0
\end{array}$$

4. If

$$A = \begin{bmatrix} 1 & 9 & 3 \\ 9 & 2 & 3 \\ 3 & 3 & 4 \end{bmatrix}$$

a. Find an elementry matrix M such that MAM^{-1} is upper Hessenberg.

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \qquad MAM^{-1} = \begin{pmatrix} 1 & 10 & 3 \\ 9 & 3 & 3 \\ 0 & \frac{10}{3} & 3 \end{pmatrix}$$

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b. The original A is symmetric; is MAM^{-1} still symmetric, and therefore tridiagonal? Show that if Q is orthogonal, QAQ^{-1} would still be symmetric. no

$$B = QAQ^{-1} = QAQ^T$$

$$B^T = QA^TQ^T = QAQ^T = B$$

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5. If

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

Do one complete LR iteration on A.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

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$$M_2 M_1 A \, M_1^{-1} M_2^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 5 & 1 \\ 0 & -16 & -4 \end{bmatrix}$$

$$M_2 M_1 A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

6. Find the eigenvalues of the quasitriangular matrix:

$$A = \begin{bmatrix} 0 & -1 & 5 & -7 \\ 1 & 0 & 5 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

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$$\det \begin{bmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \qquad \lambda = i, -i$$

$$\det \begin{bmatrix} 1-\lambda & 4 \\ 4 & 4-\lambda \end{bmatrix} = \lambda^2 - 5\lambda - 12 = 0 \qquad \lambda = \frac{5 \pm \sqrt{73}}{2}$$

$$= 6{,}772 \quad -1{,}772$$

7. If the power method applied to $(A - pI)^{-1}$ finds a largest eigenvalue of μ, what is the eigenvalue of A closest to p?

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$p + \frac{1}{\mu}$

8. What is the order of work for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

Zero one element

a. One iteration of the Jacobi method to find the eigenvalues of a symmetric matrix A. $O(N)$

b. Solution of $Ax = b$ using Gaussian elimination, if A is upper Hessenberg. $O(N^2)$

c. One QR iteration, if A is symmetric and tridiagonal. $O(N)$

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d. One Simplex step, for solving max $c^T x$ with $Ax \leq b, x \geq 0$, where A is M by N. $O(MN)$ (assume $N >> M$)

e. Solution of $Ax = b$ if an LU decomposition is known. $O(N^2)$

f. One iteration of the inverse power method, for finding the smallest eigenvalue of tridiagonal matrix A. $O(N)$

g. Solution of $Ax = b$ using Gaussian elimination, if A is banded, with bandwidth $\sqrt{N}$. $O(N^2)$

h. Reduction of a full N by N matrix to a similar upper Hessenberg matrix, by pre and post-multiplications by orthogonal Givens matrices. $O(N^3)$