Math 5330, Test I (a)

Name Key

1. If

$$A = \begin{bmatrix} 0 & 3 & 1 \\ -4 & 2 & 1 \\ 8 & 2 & 3 \end{bmatrix}$$

4

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $A = PLU$.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} -4 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

2. An N by N band matrix has K non-zero diagonals below the main diagonal and L above. If $1 << K, L << N$, approximately how many multiplications are done:

4

a. during the forward elimination, if no pivoting is done? NKL

b. during the forward elimination, if partial pivoting is done? $NK(L+K)$

c. during back substitution, if no pivoting is done? NL

d. during back substitution, if partial pivoting is done? $N(L+K)$

$$r_i \equiv \sum_{j<i} \left|\frac{a_{ij}}{a_{ii}}\right| \qquad s_i \equiv \sum_{j>i} \left|\frac{a_{ij}}{a_{ii}}\right|$$

$$r_i + s_i < 1 \quad \text{all } i$$

$$\vec{e}^{\,n} \equiv \vec{x}^{\,n} - \vec{x}$$

3. a. Prove that the Jacobi method:

4

$$x_i^{n+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j\neq i} a_{ij}x_j^n\right)$$

converges, if A is diagonal dominant.

$$|e_i^{n+1}| = \left|-\sum_{j\neq i} \frac{a_{ij}}{a_{ii}} e_j^n\right| \leq \|e^n\|_\infty (r_i + s_i)$$

suppose $\|e^{n+1}\|_\infty = |e_m^{n+1}|$

$$\|e^{n+1}\|_\infty \leq \|e^n\|_\infty (r_m + s_m) < \|e^n\|_\infty$$

b. Prove that the Gauss-Seidel method:

$$x_i^{n+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{n+1} - \sum_{j>i} a_{ij}x_j^n\right).$$

converges, if A is diagonal dominant.

4

$$|e_i^{n+1}| = \left|-\sum_{j<i} \frac{a_{ij}}{a_{ii}} e_j^{n+1} - \sum_{j>i} \frac{a_{ij}}{a_{ii}} e_j^n\right|$$

$$\leq r_i \|e^{n+1}\|_\infty + s_i \|e^n\|_\infty$$

$$\|e^{n+1}\|_\infty \leq r_m \|e^{n+1}\|_\infty + s_m \|e^n\|_\infty$$

$$\|e^{n+1}\|_\infty \leq \frac{s_m}{1-r_m} \|e^n\|_\infty < \|e^n\|_\infty$$

4. Which of the ~~following~~ number 2 linear systems would you expect to produce the most relative round-off error, using Gauss elimination with partial pivoting? Justify your answer.

4

$$\begin{bmatrix} 10^{-9} & 10^{-8} \\ 10^{-8} & 10^{-9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$* \rightarrow \begin{bmatrix} 1000 & 1001 \\ -999 & -1000 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \text{second largest cond \#}$$

$$\begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \text{largest cond \#, but diagonal}$$

5. Define:

5

a. orthogonal matrix

$AA^T = I$

b. lower Hessenberg matrix

$A_{ij} = 0$ for $j > i+1$

c. positive definite matrix

$A = A^T$ and all eigenvalues positive

d. $\|x\|_p$, if x is a vector and $1 \le p < \infty$

$$\|x\|_p = \left[\sum_{i=1}^{N} x_i^p \right]^{\frac{1}{p}}$$

e. $\|A\|_p$, if A is a matrix

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

6. The following Fortran program solves a linear system $Ax = b$ with symmetric matrix A, using Gauss-Jordan without pivoting, but taking advantage of symmetry. For large N, approximately how many multiplications are done? Show your work.

S

```
          SUBROUTINE DLINEQ(A,N,X,B)
          DOUBLE PRECISION A(N,N),X(N),B(N),LJI
C                        REDUCTION TO DIAGONAL
          DO 50 I=1,N
C                        ELIMINATE ELEMENTS ABOVE DIAGONAL IN COLUMN I
             DO 20 J=1,I-1
                LJI = A(J,I)/A(I,I)
                DO 10 K=I,N
                   A(J,K) = A(J,K) - LJI*A(I,K)
(I)   10        CONTINUE
                B(J) = B(J) - LJI*B(I)
      20     CONTINUE
C                        ELIMINATE ELEMENTS BELOW DIAGONAL IN COLUMN I.
C                        TAKE ADVANTAGE OF SYMMETRY HERE.
             DO 40 J=I+1,N
                LJI = A(I,J)/A(I,I)
                DO 30 K=J,N
                   A(J,K) = A(J,K) - LJI*A(I,K)
(II)  30        CONTINUE
                B(J) = B(J) - LJI*B(I)
      40     CONTINUE
      50  CONTINUE
C                        SOLVE DIAGONAL SYSTEM
          DO 55 I=1,N
             X(I) = B(I)/A(I,I)
      55  CONTINUE
          RETURN
          END
```

(I) $\displaystyle\sum_{i=1}^{N} I(N-I) = N\frac{1}{2}N^2 - \frac{1}{3}N^3 = \frac{1}{6}N^3$

(II) $\displaystyle\sum_{i=1}^{N}\sum_{j=i}^{N} (N-J) = \sum_{i=1}^{N}\frac{1}{2}(N-I)^2 = \frac{1}{2}\sum_{i=1}^{N} I^2 = \frac{1}{6}N^3$

total $= \frac{1}{3}N^3$

Math 5330, Test I (B)

Name Key

1. Find the LU decomposition (no pivoting) of

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -3 & 5 \\ 3 & 9 & -4 \end{bmatrix}$$

3

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

2. A MATLAB program to solve a symmetric system $Ax = b$ does most of its work in the loops:

3

```
for I=1:N-1
   for J=I+1:N
      for K=J:N
         A(J,K) = A(J,K) - LJI*A(I,K)
      end
   end
end
```

For large N, approximately how many multiplications are done (show work)?

$(L \equiv N-J)$

$$\sum_{I=1}^{N-1} \sum_{J=I+1}^{N} (N-J) \cong \sum_{I=1}^{N} \sum_{L=0}^{N-I-1} L$$

$$\cong \sum_{I=1}^{N} \frac{(N-I)^2}{2} \quad \cong \sum_{M=0}^{N-1} \frac{M^2}{2} \cong \frac{1}{6} N^3$$

1

$M = N-I$

3. Prove that $\frac{||\Delta x||}{||x||} \leq cond(A)\frac{||\Delta b||}{||b||}$ if $Ax = b$ and $A(x + \Delta x) = b + \Delta b$.

$A \Delta x = \Delta b$

$||\Delta x|| = ||A^{-1}\Delta b|| \leq ||A^{-1}|| \, ||\Delta b||$

3 $\quad ||b|| = ||Ax|| \leq ||A|| \, ||x||$ $\longrightarrow$ $\frac{||\Delta x||}{||x||} \leq ||A|| \, ||A^{-1}|| \frac{||\Delta b||}{||b||}$

$\frac{1}{||x||} \leq \frac{||A||}{||b||}$

4. If we use the usual finite difference approximation, the DE $u''(x) = f(x), u(0) = u(\pi) = 0$ becomes:

$$U_{i+1} - 2U_i + U_{i-1} = h^2 f(x_i), \quad i = 1, ..., N-1$$
$$U(x_0) = U(x_N) = 0$$

where $h = \pi/N, x_i = ih, U_i \approx u(x_i)$.

a. This is a linear system of $N-1$ equations for the $N-1$ unknowns $U_1, ..., U_{N-1}$. If a band solver is used to solve the system, the work is proportional to what power of N? N^1

1

b. If Jacobi's iterative method is used to solve it, the iteration will take the form $U^{k+1} = BU^k + c$; what is the matrix B?

2 $\quad B = \begin{bmatrix} 0 & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & & \ddots & \end{bmatrix}$

c. What are the eigenvalues of the B matrix (hint: for any $m = 1, ..., N-1$, the vector U with components $U_i = sin(mx_i)$ is an eigenvector. You will need the trig identity $sin(a+b) = sin(a)cos(b) + cos(a)sin(b)$)

2

$$(BU)_i = \frac{1}{2}(U_{i+1} + U_{i-1}) = \frac{1}{2}[\sin m(x_i+h) + \sin m(x_i-h)]$$

$$= \sin m x_i \cos mh \qquad \lambda_m = \cos mh$$

d. What is the largest eigenvalue of B in absolute value? Will the Jacobi method converge?

1

$$|\lambda_1| = |\lambda_{N-1}| = |\cos h| < 1 \qquad \text{so yes}$$

e. Given that the error goes down each iteration by a factor approximately equal to the largest eigenvalue, estimate how many iterations of the Jacobi method are required to decrease the error by a factor of ϵ. (Hint: $cos(z) \approx 1 - z^2/2$ and $ln(1+z) \approx z$ for $z \approx 0$)

2

$$(\cos h)^M = \epsilon$$

$$M \ln(\cos h) = \ln \epsilon$$

$$M \ln\left(1 - \frac{h^2}{2}\right) \cong \ln \epsilon$$

$$M\left(-\frac{h^2}{2}\right) \cong \ln \epsilon$$

$$M \cong \frac{2}{h^2} \ln\left(\frac{1}{\epsilon}\right)$$

$$= \frac{2}{\pi^2} N^2 \ln\left(\frac{1}{\epsilon}\right)$$

f. The total work to solve the linear system using the Jacobi iterative method is then proportional to what power of N? Which is faster for this tridiagonal system–a band solver or the Jacobi iterative method?

1

N^3 band solver faster

g. If the Gauss-Seidel iterative method is used to solve the linear system, what is the matrix B (see part (b)) now? You need not write the matrix out explicitly, for example, you can write it as $E^{-1}F$, where you define E and F. Gauss-Seidel will converge if and only if what is true about B?

2

$$u_i^{k+1} = \tfrac{1}{2} u_{i-1}^{k+1} + \tfrac{1}{2} u_{i+1}^{k} + c_i$$

$$\underbrace{\begin{pmatrix} 1 & & & \\ -\frac{1}{2} & 1 & & \\ & -\frac{1}{2} & 1 & \\ & & & \ddots \end{pmatrix}}_{= E} u^{k+1} = \underbrace{\begin{pmatrix} 0 & \frac{1}{2} & & \\ & 0 & \frac{1}{2} & \\ & & 0 & \frac{1}{2} \\ & & & \ddots \end{pmatrix}}_{= F} u^{k} + \vec{c}$$

$$B = E^{-1} F$$

Converges $\Leftrightarrow$ all eigenvalues of B less than one in absolute value

Math 5330, Test I (c)

Name Key

1. a. Show that any matrix which has a "Cholesky" decomposition $A = LL^T$, with L nonsingular, is positive definite, that is, show it is symmetric and $x^T Ax > 0$ for any nonzero vector x.

$$A^T = (LL^T)^T = LL^T = A \quad \text{symmetric}$$

$$x^T A x = x^T L L^T x = (L^T x)^T L^T x = \|L^T x\|^2 > 0$$

unless $L^T x = 0 \Rightarrow x = 0$

b. Show that

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

is positive definite, by finding its LU decomposition.

$$L = U^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \text{so } A = LU = LL^T$$

2. An N by N band matrix has $N^{1/3}$ non-zero diagonals below the main diagonal and the same number above. If N is large, approximately how many multiplications are done:

a. during the forward elimination, if no pivoting is done? $N(N^{1/3})^2 = N^{5/3}$

b. during the forward elimination, if partial pivoting is done? $2N^{5/3}$

c. during back substitution, if no pivoting is done? $N N^{1/3} = N^{4/3}$

d. during back substitution, if partial pivoting is done? $2N^{4/3}$

3. A MATLAB program which solves a symmetric linear system, with no pivoting, does most of its work in the loops:

```
for I=1:N-1
   for J=I+1:N
      for K=J:N
         A(J,K) = A(J,K) - LJI*A(I,K)
      end
   end
end
```

Approximately how many multiplications are done (show work)? How does this compare to Gaussian elimination for a nonsymmetric system?

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(N-j+1) = \sum_{i=1}^{N-1}\left[(N-i)\ldots 1\right] \cong \sum_{i=1}^{N-1}\frac{(N-i)^2}{2} = \sum_{l=N-1}^{1}\frac{l^2}{2}$$

$$\cong \frac{1}{2}\sum_{l=1}^{N} l^2 = \frac{1}{6}N^3$$

(50% less)

4. a. If a matrix is decomposed into its (strictly) subdiagonal, diagonal, and (strictly) superdiagonal parts, $A = L + D + U$, the Jacobi iterative method for solving $Ax = b$ will converge if and only if all eigenvalues of what matrix are less than 1 in absolute value?

$$Dx_{n+1} = (-L-U)x_n + b \qquad -D^{-1}(L+U)$$

b. Same question, for the Gauss-Seidel method.

$$Dx_{n+1} = -Lx_{n+1} - Ux_n + b \qquad -(D+L)^{-1}U$$

c. Using parts [a.] and [b.], show that both Jacobi and Gauss-Seidel methods will converge if A is either upper triangular or lower triangular, and all its diagonal elements ~~all~~ are nonzero. (Hint: the eigenvalues of an upper or lower triangular matrix are its diagonal entries.)

if $L = 0 \Rightarrow -D^{-1}U$ and $-D^{-1}U$ both have 0's on diagonal, so eigenvalues all 0

$U = 0 \Rightarrow -D^{-1}L$ and 0 have 0's on diagonal so eigenvalues all 0.

2

5. Approximately how many significant digits would you expect in the solution of $Ax = b$ if Gaussian elimination with partial pivoting is used on a computer with machine precision $\epsilon = 10^{-12}$, and

about 6

$$A = \begin{bmatrix} 1.000001 & 1 \\ 1 & 1 \end{bmatrix} \qquad A^{-1} = \frac{\begin{bmatrix} 1 & -1 \\ -1 & 1.000001 \end{bmatrix}}{(.000001)}$$

$$\text{cond}(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

$$= 2 \frac{2}{.000001} = 4 \cdot 10^{+6} \qquad \frac{\|\delta x\|}{\|x\|} \approx 4 \cdot 10^{+6} (10^{-12}) \doteq 4 \cdot 10^{-6}$$

6. Define:

a. orthogonal matrix

$$A^T A = I$$

b. lower Hessenberg matrix

$$A_{ij} = 0 \quad \text{for } j > i+1$$

c. permutation matrix

rows are permutation of rows of I

d. $\|x\|_p$, if x is a vector and $1 \le p < \infty$

$$\|x\|_p = \left[\sum_{i=1}^{N} |x_i|^p \right]^{\frac{1}{p}}$$

e. $\|A\|_p$, if A is a matrix

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Math 5330, Test I (d)

Name Key

Work any 5 problems, clearly indicate which problem is NOT to be graded.

1. If

$$A = \begin{bmatrix} -4 & 2 & 1 \\ 0 & 3 & 1 \\ 8 & 2 & 3 \end{bmatrix}$$

find a lower triangular matrix L, and an upper triangular matrix U such that $A = LU$.

5

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} -4 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}}_{U}$$

2. An N by N band matrix has $\sqrt{N}$ non-zero diagonals below the main diagonal and the same number above. If N is large, approximately how many multiplications are done:

4

a. during the forward elimination, if no pivoting is done? $N(\sqrt{N})^2 = N^2$

b. during the forward elimination, if partial pivoting is done? $2N(\sqrt{N})^2 = 2N^2$

c. during back substitution, if no pivoting is done? $N\sqrt{N}$

d. during back substitution, if partial pivoting is done? $2N\sqrt{N}$

S

3. A MATLAB program which solves a symmetric linear system, with no pivoting, using Gauss-Jordan, does most of its work in the loops:

$$\sum_{I=1}^{N} I(N-I)$$

$+$

$$\sum_{I=1}^{N}\sum_{J=I}^{N}(N-J)$$

```
      for I=1:N
%                         ELIMINATE ELEMENTS ABOVE DIAGONAL IN COLUMN I
         for J=1:I-1
            for K=I:N
               A(J,K) = A(J,K) - LJI*A(I,K);
            end
         end
%                         ELIMINATE ELEMENTS BELOW DIAGONAL IN COLUMN I.
%                         TAKE ADVANTAGE OF SYMMETRY HERE.
         for J=I+1:N
            for K=J:N
               A(J,K) = A(J,K) - LJI*A(I,K)
            end
         end
      end
```

Approximately how many multiplications are done (show work)? How does this compare to Gaussian elimination for a nonsymmetric system?

$$= N\sum_{I=1}^{N} I - \sum_{I=1}^{N} I^2 + \sum_{I=1}^{N}\left(1+2+\dots(N-I)\right) \doteq \frac{1}{2}N^3 - \frac{1}{3}N^3$$

$$+ \sum_{I=1}^{N}\frac{(N-I)^2}{2} = \frac{1}{6}N^3 + \quad + \frac{1}{2}\sum_{I=1}^{N} I^2$$

$$= \frac{1}{3}N^3$$

Same as GE

4. Prove that $\frac{\|\Delta x\|}{\|x\|} \leq cond(A)\frac{\|\Delta b\|}{\|b\|}$ if $Ax = b$ and $A(x + \Delta x) = b + \Delta b$.

S

$A\Delta x = \Delta b$

$\|\Delta x\| \leq \|A^{-1}\|\|\Delta b\|$

$Ax = b$

$\|b\| \leq \|A\|\|x\|$

$\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$

$\rightarrow \frac{\|\Delta x\|}{\|x\|} \leq \underbrace{\|A\|\|A^{-1}\|}_{cond(A)}\frac{\|\Delta b\|}{\|b\|}$

5. Would you expect the Jacobi iterative method to converge, when used to solve:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

5

What about the Gauss-Seidel method? Justify your answers theoretically, that is, without actually taking any iterations.

$$GS = \text{Jacobi} = \begin{pmatrix} x^{n+1} \\ y^{n+1} \\ z^{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -2 & -3 \\ 0 & 0 & -\frac{5}{4} \\ 0 & 0 & 0 \end{pmatrix}}_{B} \begin{pmatrix} x^n \\ y^n \\ z^n \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{6} \end{pmatrix}$$

eigenvalues of B all $= 0$ so converges

6. Define:

6

a. orthogonal matrix $A^{-1} = A^T$

b. lower Hessenberg matrix $A_{ij} = 0$ if $j > i+1$

c. tridiagonal matrix $A_{ij} = 0$ if $|i-j| > 1$

d. positive definite matrix $A^T = A$, all eigenvalues positive

e. $\|x\|_1$, if x is a vector $\sum_{i=1}^{N} |x_i|$

f. $\|A\|_2$, if A is a matrix $\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$ or $\sqrt{\lambda_{max}(A^TA)}$

Math 5330, Test I (e)

Name Key

1. If

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $PA = LU$.

4

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Prove that $\frac{\|\Delta x\|}{\|x\|} \leq cond(A)\frac{\|\Delta b\|}{\|b\|}$ if $Ax = b$ and $A(x + \Delta x) = b + \Delta b$.

4

$$A\Delta x = \Delta b$$

$$\Delta x = A^{-1}\Delta b$$

$$\|\Delta x\| \leq \|A^{-1}\|\|\Delta b\|$$

$$Ax = b$$

$$\|b\| \leq \|A\|\|x\|$$

$$\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$$

$$\frac{\|\Delta x\|}{\|x\|} \leq \underbrace{\|A^{-1}\|\|A\|}_{cond(A)}\frac{\|\Delta b\|}{\|b\|}$$

3. If we use the usual finite difference approximation, the DE $u''(x) = f(x), u(0) = u(\pi) = 0$ becomes:

$$U_{i+1} - 2U_i + U_{i-1} = h^2 f(x_i), \quad i = 1, ..., N-1$$
$$U(x_0) = U(x_N) = 0$$

where $h = \pi/N, x_i = ih, U_i \approx u(x_i)$.

2 a. This is a linear system of $N-1$ equations for the $N-1$ unknowns $U_1, ..., U_{N-1}$. If a band solver is used to solve the system, the work is proportional to what power of N?

$O(N)$

2 b. If Jacobi's iterative method is used to solve it, the iteration will take the form $U^{k+1} = BU^k + c$; what is the matrix B?

$$B = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \ddots \\ & & \ddots & & \end{bmatrix}$$

2 c. What is $\|B\|_\infty$?

1

2 d. What are the eigenvalues of the B matrix (hint: for any $m = 1, ..., N-1$, the vector U with components $U_i = sin(mx_i)$ is an eigenvector. You will need the trig identity $sin(a+b) = sin(a)cos(b) + cos(a)sin(b)$)

$$(BU)_i = \frac{1}{2}\sin(mx_{i-1}) + \frac{1}{2}\sin(mx_{i+1}) = \frac{1}{2}\sin(mx - mh) + \frac{1}{2}\sin(mx + mh)$$
$$= \frac{1}{2}\left[\sin mx \cos mh - \cancel{\cos mx \sin mh} + \sin mx \cos mh + \cancel{\cos mx \sin mh}\right]$$
$$= \cos mh \sin mx$$
$$= \cos mh \, U_i$$
$$\lambda_m = \cos mh$$

2 e. What is the largest eigenvalue of B in absolute value? Will the Jacobi method converge?

$\lambda_{max} = \lambda_1 = \cos h$ yes

4. Which of the following linear systems has the largest condition number? Would you expect to have serious round-off error problems if you solved this system, using Gauss elimination with partial pivoting?

$$\begin{bmatrix} 1 & 10^{-9} \\ 10^{-9} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ cond(A) ≈ 1

$$\begin{bmatrix} 1.000001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ cond(A) $\approx 4 \cdot 10^6$

3

$$\begin{bmatrix} 10^{-10} & 0 \\ 0 & 10^{10} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ cond(A) $\approx 10^{20}$

but no problem w. roundoff

5. Define:

5

a. orthogonal matrix

$A^TA = I$

b. upper Hessenberg matrix

$A_{ij} = 0$ for $j < i-1$

c. positive definite matrix

A is symmetric and all eigenvalues > 0

d. $\|x\|_p$, if x is a vector and $1 \le p < \infty$

$\|x\|_p = \left(\sum |x_i|^p \right)^{\frac{1}{p}}$

e. $\|A\|_p$, if A is a matrix

$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$

6. The following MATLAB program solves a linear system $Ax = b$ with no pivoting, it knocks out all elements above and below the diagonal and then solves the final diagonal system. However, unlike Gauss-Jordan, it knocks out all elements below the diagonal "before" knocking out the elements above. For large N, approximately how many multiplications are done? Show your work.

```
      function X = DLINEQ(A,N,B)
%                                  REDUCE TO UPPER TRIANGULAR FORM (NO PIVOTING)
      for I=1:N-1
%                                  KNOCK OUT ELEMENTS BELOW DIAGONAL IN COLUMN I
         for J=I+1:N
            LJI = A(J,I)/A(I,I);
            for K=I:N
               A(J,K) = A(J,K) - LJI*A(I,K);
            end
            B(J) = B(J) - LJI*B(I);
         end
      end
%                                  NOW REDUCE TO DIAGONAL FORM
      for I=N:-1:2
%                                  KNOCK OUT ELEMENTS ABOVE DIAGONAL IN COLUMN I
         for J=1:I-1
            LJI = A(J,I)/A(I,I);
            A(J,I) = A(J,I) - LJI*A(I,I);
            B(J) = B(J) - LJI*B(I);
         end
      end
%                                  NOW SOLVE DIAGONAL SYSTEM
      for I=1:N
         X(I) = B(I)/A(I,I);
      end
```

$$\sum_{i=1}^{N-1} (N-i)(N-i+1) \simeq \sum_{k=1}^{N} k^2 = \frac{1}{3}N^3$$

4 $\quad O(N^2)$

$O(N)$

total $\simeq \frac{1}{3}N^3$, same as GE

Math 5330, Test I (f)

Name ___Key___

1. If

$$A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 0 & 1 \\ -4 & -4 & 4 \end{bmatrix}$$

do Gaussian elimination *with partial pivoting* to find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $A = PLU$.

IPERM

5

$$\begin{array}{c} 3 \\ 2 \\ 1 \end{array}\begin{bmatrix} -4 & -4 & 4 \\ 1 & 0 & 1 \\ 3 & -2 & 2 \end{bmatrix} \rightarrow \begin{array}{c} 3 \\ 2 \\ 1 \end{array}\begin{bmatrix} -4 & -4 & 4 \\ (-\frac{1}{4}) & -1 & 2 \\ (-\frac{3}{4}) & -5 & 5 \end{bmatrix} \rightarrow \begin{array}{c} 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} -4 & -4 & 4 \\ (-\frac{3}{4}) & -5 & 5 \\ (-\frac{1}{4}) & -1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} -4 & -4 & 4 \\ (-\frac{3}{4}) & -5 & 5 \\ (-\frac{1}{4}) & (\frac{1}{5}) & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ -\frac{1}{4} & \frac{1}{5} & 1 \end{bmatrix}\begin{bmatrix} -4 & -4 & 4 \\ 0 & -5 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Prove that the Jacobi method:

$$x_i^{n+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j \neq i} a_{ij} x_j^n\right)$$

converges, if A is diagonal dominant ($|a_{ii}| > \sum_{j \neq i} |a_{ij}|$, for each i).

5

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j \neq i} a_{ij} x_j\right)$$

$$|e_i^{n+1}| = \left|\frac{1}{a_{ii}}\left(-\sum_{j \neq i} a_{ij} e_j^n\right)\right| \le \frac{1}{|a_{ii}|} \|e^n\|_\infty \sum_{j \neq i} |a_{ij}|$$

$$|e_i^{n+1}| < \|e^n\|_\infty \text{ all } i \Rightarrow \|e^{n+1}\|_\infty < \|e^n\|_\infty$$

3. What is the order of work ($O(N^\alpha)$) for each of the following? Assume all matrices are N by N, where N is large, and that advantage is taken of any special structure mentioned. Assume A is full, for parts a,b,c,d.

   $O(N^3)$ a. The forward elimination stage of Gaussian elimination applied to $Ax = b$.

   $O(N^2)$ b. The backward substitution stage of Gaussian elimination.

   $O(N^2)$ c. Solution of $Ax = b$ if an LU decomposition is known.

   $O(N^3)$ d. The Gauss-Seidel iteration to solve $Ax = b$, if N iterations are required for convergence.

   8 $O(N)$ e. Solution of $Ax = b$ if A is tridiagonal, except that A_{1N} and A_{N1} are also nonzero.

   $O(N^2)$ f. Solution of $Ax = b$ using Gaussian elimination if A is banded, with bandwidth $\sqrt{N}$, and no pivoting is done.

   $O(N^2)$ g. Same as (f) but now partial pivoting is done.

   $O(N^{1.5})$ h. Same as (f) but now assume an LU decomposition of A is already known.

4. Which of the following linear systems would you expect to produce the most relative round-off error, using Gauss elimination with partial pivoting? Justify your answer.

$$\begin{bmatrix} 10^{-9} & 10^{-8} \\ 10^{-8} & 10^{-9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2

$$\begin{bmatrix} 1 & 1.00001 \\ -0.99999 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. A MATLAB program which solves a linear system using Gauss-Jordan does most of its work in the loops:

```
for I=1:N
   for J=1:N
      if (J ~= I)
         for K=I:N
            A(J,K) = A(J,K) - LJI*A(I,K);
         end
      end
   end
end
```

S

Approximately how many multiplications are done (show work)? How does this compare to Gaussian elimination?

$$\sum_{i=1}^{N} N(N-I) = N^2 \sum_{i=1}^{N} - N \sum_{i=1}^{N} I = N^3 - N\left(\tfrac{1}{2}N^2\right) \simeq \tfrac{1}{2}N^3$$

6. Would you expect the Jacobi iterative method to converge, when used to solve:

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

S

What about the Gauss-Seidel method? Justify your answers theoretically, that is, without actually taking any iterations.

Jacobi = GS $\rightarrow$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}^{n+1} = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1.25 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^n \\ y^n \\ z^n \end{pmatrix} + \begin{pmatrix} 1 \\ 0.25 \\ 0.1666 \end{pmatrix}$

eigenvalues all 0, so (both) converge

Math 5330, Test I (g)

Name Key

3

1. If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 8 \\ -2 & -5 & 4 \end{bmatrix}$$

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $PA = LU$.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 10 \\ 0 & 0 & -1 \end{bmatrix}$$

3

2. Use the decomposition of A from Problem 1 to solve $Ax = b$, where $b = (6, 17, -3)$. That is, multiply both sides by P : $PAx = Pb$ so $LUx = Pb$, then solve $Ly = Pb$, then $Ux = y$.

$$Pb = \begin{pmatrix} 6 \\ -3 \\ 17 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 17 \end{pmatrix} \rightarrow \begin{pmatrix} 6 \\ 9 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 10 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

3. An N by N band matrix has L non-zero diagonals below the main diagonal and L above. If $1 << L << N$, approximately how many multiplications are done:

   a. during the forward elimination, if no pivoting is done? NL^2
   b. during the forward elimination, if partial pivoting is done? $2NL^2$
   c. during back substitution, if no pivoting is done? NL
   d. during back substitution, if partial pivoting is done? $2NL$

4

Hint: below is a MATLAB program which solves a banded linear system with no pivoting. How do the limits change if partial pivoting is done?

```
      function X = LBANDO(A,B,N,L)
%
%     ARGUMENT DESCRIPTIONS
%
%     A - (INPUT) A IS AN N BY 2*L+1 ARRAY CONTAINING THE BAND MATRIX.
%         A(I,L+1+J-I) CONTAINS THE MATRIX ELEMENT IN ROW I, COLUMN J.
%     X - (OUTPUT) X IS THE SOLUTION VECTOR OF LENGTH N.
%     B - (INPUT) B IS THE RIGHT HAND SIDE VECTOR OF LENGTH N.
%     N - (INPUT) N IS THE NUMBER OF EQUATIONS AND NUMBER OF UNKNOWNS
%         IN THE LINEAR SYSTEM.
%     L - (INPUT) L IS THE HALF-BANDWIDTH, DEFINED AS THE MAXIMUM VALUE
%         OF ABS(I-J) SUCH THAT AIJ IS NONZERO.
%
      MD = L+1;
%                              BEGIN FORWARD ELIMINATION
      for K=1:N-1
         if (A(K,MD) == 0.0)
            error ('Zero pivot encountered')
         end
         for I=K+1:min(K+L,N)
            AMUL = -A(I,MD+K-I)/A(K,MD);
            if (AMUL ~= 0.0)
%                              ADD AMUL TIMES ROW K TO ROW I
               for J=K:min(K+L,N)
                  A(I,MD+J-I) = A(I,MD+J-I) + AMUL*A(K,MD+J-K);
               end
               B(I) = B(I) + AMUL*B(K);
            end
         end
```

```
      end
      if (A(N,MD) == 0.0)
         error ('Zero pivot encountered')
      end
%                                    BEGIN BACK SUBSTITUTION
      X(N) = B(N)/A(N,MD);
      for K=N-1:-1:1
         SUM = 0;
         for J=K+1:min(K+L,N)
            SUM = SUM + A(K,MD+J-K)*X(J);
         end
         X(K) = (B(K)-SUM)/A(K,MD);
      end
```

4. Find the condition number of

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1000 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1000 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2 $\quad \mathrm{Cond}(B) = \|B\| \|B^{-1}\| = (1001)(1001) \cong 10^6$

5. Approximately how many multiplications does the following MATLAB program do, for large N:

```
      X = 1;
      for L=1:N
         for I=L:N
            for J=L:N
               for K=L:N
                  X = X*X;
               end
            end
         end
      end
```

2 $\quad N^3 + (N-1)^3 + \dots 2^3 + 1^3 \cong \frac{1}{4} N^4$

6. Consider the 1D boundary value problem $-U_{xx} + U = sin(x)$ with boundary conditions $U(0) = 1, U(2\pi) = 2$. This differential equation can be approximated using the finite difference equation:

$$\frac{-U_{i+1} + 2U_i - U_{i-1}}{h^2} + U_i = sin(x_i)$$

for $i = 2, ..., n$, where $x_i = (i-1)h, h = 2\pi/n$, and U_i is an approximation to $U(x_i)$. The boundary conditions imply $U_1 = 1, U_{n+1} = 2$.

The MATLAB program below should do 1000 iterations of the Gauss-Seidel iteration to solve this finite difference system, complete the incomplete statement. Explain why convergence is guaranteed. A is diagonal dominant

3

```
n = 20;
h = 2*pi/n;
u(2:n) = 0;
u(1) = 1;
u(n+1) = 2;
for iter=1:1000
   for i=2:n
% complete this statement:
      u(i) = (h^2* sin((i-1)*h) + u(i-1) + u(i+1)) / (2+h^2);
   end
end
```

7. Define:

a. orthogonal matrix

$A^TA = I$

4

b. upper Hessenberg matrix

$a_{ij} = 0$ for $j < i-1$

c. positive definite matrix

A^TA and $x^TAx > 0$ for all $x \neq 0$

d. $\|A\|_p$, if A is a matrix

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Math 5330, Test I (h)

Name Key

1. If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ -2 & -5 & -4 \end{bmatrix}$$

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $PA = LU$.

3

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

2. Use the decomposition of A from Problem 1 to solve $Ax = b$, where $b = (8, 18, -16)$. That is, multiply both sides by P : $PAx = Pb$ so $LUx = Pb$, then solve $Ly = Pb$, then $Ux = y$.

2

$$Pb = \begin{pmatrix} 8 \\ -16 \\ 18 \end{pmatrix}$$

$$Ly = \begin{pmatrix} 8 \\ -16 \\ 18 \end{pmatrix} \quad y = \begin{pmatrix} 8 \\ 0 \\ 2 \end{pmatrix}$$

$$Ux = \begin{pmatrix} 8 \\ 0 \\ 2 \end{pmatrix} \quad x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

4

3. Consider the linear system:

$$\begin{bmatrix} -4 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

a. If the Jacobi iterative method is written in the form $x^{n+1} = Bx^n + c$, what is B?

$$B = \begin{pmatrix} 0 & +\frac{5}{4} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

b. Determine *theoretically* if the Jacobi method will converge, without doing any actual iterations.

$\lambda^2 + \frac{5}{8} = 0 \quad \lambda = \pm\sqrt{\frac{5}{8}}\, i$ converge

c. If the Gauss-Seidel method is written in the form $x^{n+1} = Bx^n + c$, what is B?

$$B = \begin{pmatrix} 0 & \frac{5}{4} \\ 0 & -\frac{5}{8} \end{pmatrix}$$

d. Determine *theoretically* if the Gauss-Seidel method will converge, without doing any actual iterations.

$\lambda = 0, -\frac{5}{8}$ converges

4. Approximately how many multiplications does the following MATLAB program do, for large N?

```
X = 1;
for L=1:N
   for I=1:L
      for J=1:I
        for K=1:J
           X = X*X;
        end
      end
   end
end
```

3

$$\sum_{L=1}^{N}\sum_{I=1}^{L}\sum_{J=1}^{I} J \doteq \sum_{L=1}^{N}\sum_{I=1}^{L} 1+2+\ldots I$$

$$\simeq \sum_{L=1}^{N}\sum_{I=1}^{L}\frac{I^2}{2} \doteq \frac{1}{2}\sum_{L=1}^{N} 1^2+2^2+\ldots L^2 \cong \frac{1}{2}\frac{1}{3}\sum_{L=1}^{N} L^3$$

$$\doteq \frac{1}{24}N^4$$

5. Compute the condition number (infinity norm) for each of these matrices and tell which you would expect to produce the most relative round-off error, using Gauss elimination with partial pivoting?

$$\overset{A}{\begin{bmatrix} 1000 & 1001 \\ -999 & -1000 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1000 & 1001 \\ -999 & -1000 \end{bmatrix}$$

2

$$\overset{B}{\begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{9} \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 10^{9} & 0 \\ 0 & 10^{-9} \end{bmatrix}$$

$\text{cond}(A) = (2001)^2 \cong 4\cdot 10^6 \Leftarrow$ more r.o. error

$\text{cond}(B) = 10^{18}$

6

6. Define:

a. orthogonal matrix

$$QQ^T = I$$

b. lower Hessenberg matrix

$$A_{ij} = 0 \quad \text{for} \quad j > i+1$$

c. tridiagonal matrix

$$A_{ij} = 0 \quad \text{for} \quad |i-j| > 1$$

d. positive definite matrix

$A^T = A$ and all eigenvalues positive

or ($x^T A x > 0$ all $x \neq 0$)

e. $\|x\|_\infty$, if x is a vector

$$\max_i |x_i|$$

f. $\|A\|_p$, if A is a matrix

$$\max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Math 5330 Exam I (1)

Name Key

1. a. If

$$A = \begin{bmatrix} 0 & 3 & 1 \\ -4 & 2 & 1 \\ 8 & 2 & 3 \end{bmatrix}$$

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $A = PLU$.

$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$ $U = \begin{pmatrix} -4 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

15

b. What is the main use for an LU decomposition of a large matrix?

5

Solving >1 system w. same matrix
$O(N^2)$ work instead of $O(N^3)$

2. A MATLAB program to solve a symmetric system $Ax = b$ does most of its work in the loops:

```
for I=1:N-1
   for J=I+1:N
      for K=J:N
         A(J,K) = A(J,K) - LJI*A(I,K)
      end
   end
end
```

10

For large N, approximately how many multiplications are done (show work)?

$$\sum_{i=1}^{N} \sum_{j=i+1}^{N} \sum_{k=j}^{N} 1 = \sum_{i=1}^{N} \sum_{j=i+1}^{N} (N-j)$$

$$= \sum_{i=1}^{N} (N-i) + \ldots 1 \simeq \sum_{i=1}^{N} \tfrac{1}{2}(N-i)^2 = \tfrac{1}{2}\left[N^2 + \ldots 1^2\right] \simeq \tfrac{1}{2}\,\tfrac{1}{3}N^3$$

$$= \tfrac{1}{6}N^3$$

3. If we use the usual finite difference approximation, the DE $u''(x) = f(x), u(0) = u(\pi) = 0$ becomes:

$$U_{i+1} - 2U_i + U_{i-1} = h^2 f(x_i), \quad i = 1, ..., N-1$$
$$U(x_0) = U(x_N) = 0$$

where $h = \pi/N, x_i = ih, U_i \approx u(x_i)$.

a. This is a linear system of $N-1$ equations for the $N-1$ unknowns $U_1, ..., U_{N-1}$. If a band solver is used to solve the system, the work is proportional to what power of N?

5

$O(N)$

b. If Jacobi's iterative method is used to solve it, the iteration will take the form $U^{k+1} = BU^k + c$; what is the matrix B?

S

$$B = \begin{bmatrix} 0 & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & & \ddots & \end{bmatrix}$$

c. What are the eigenvalues of the B matrix (hint: for any $m = 1, ..., N-1$, the vector U with components $U_i = sin(mx_i)$ is an eigenvector. You will need the trig identity $sin(a+b) = sin(a)cos(b) + cos(a)sin(b)$)

S

fixed m: $(BU)_i = \frac{1}{2} U_{i+1} + \frac{1}{2} U_{i-1} = \frac{1}{2} \sin m(x_i + h) + \frac{1}{2} \sin m(x_i - h)$

$= \sin mx_i \cos mh = \cos mh \, U_i$ $\quad \lambda_m = \cos mh \quad m = 1, .., N-1$

d. What is the largest eigenvalue of B in absolute value? Will the Jacobi method converge?

S $|\lambda_1| = |\lambda_{N-1}| = |\cos h| < 1$ so yes

e. Given that the error goes down each iteration by a factor approximately equal to the largest eigenvalue, estimate how many iterations of the Jacobi method are required to decrease the error by a factor of ϵ. (Hint: $cos(z) \approx 1 - z^2/2$ and $ln(1+z) \approx z$ for $z \approx 0$)

S

$\cos h \approx 1 - \frac{h^2}{2} \quad \left(1 - \frac{h^2}{2}\right)^N = \epsilon$

$\Rightarrow N = \frac{\ln \epsilon}{\ln\left(1 - \frac{h^2}{2}\right)} \approx \frac{\ln \epsilon}{-\frac{h^2}{2}} \approx \frac{2}{h^2} \ln\left(\frac{1}{\epsilon}\right) \approx \frac{2N^2}{\pi^2} \ln \frac{1}{\epsilon}$

5

f. The total work to solve the linear system using the Jacobi iterative method is then proportional to what power of N? Which is faster for this tridiagonal system–a band solver or the Jacobi iterative method?

$O(N^3)$ tridiagonal solver

4. a. Find a QR decomposition of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 12 \\ 0 & -5 \end{bmatrix}$$

10

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{12}{13} & \frac{5}{13} \\ 0 & -\frac{5}{13} & \frac{12}{13} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 13 \\ 0 & 0 \end{bmatrix}}_{R}$$

b. Use this QR decomposition to find $min\|Ax - b\|_2$, where $b = (1, 2, -1)$.

5

$QRx \cong b$

$Rx \cong Q^T b = \begin{pmatrix} 1 \\ 29/13 \\ -2/13 \end{pmatrix}$

$x = \begin{pmatrix} 1 \\ \frac{29}{169} \end{pmatrix}$

c. What is the main use for a QR decomposition of a large matrix?

finding $\min \|Ax-b\|$ for several b, same A

5

5. Prove that if $AA^Tz = b$, and $x = A^Tz$, then x minimizes $\|x\|_2$ over all solutions of $Ax = b$.

$Ax = AA^Tz = b$ so x is solution $(e = y - x)$

Let y be any other solution $Ay = b$, $Ae = 0$

10

$\|y\|^2 = (x+e)^T(x+e) = x^Tx + 2x^Te + e^Te$

$= \|x\|^2 + \|e\|^2 + 2\underbrace{(A^Tz)^Te}_{z^TAe = 0} = \|x\|^2 + \|e\|^2 \geq \|x\|^2$

6. For what nonzero value of α is $I - \alpha\, ww^T$ orthogonal, for a vector $w \neq 0$?

$(I - \alpha ww^T)^T(I - \alpha ww^T)$

10

$= I - 2\alpha ww^T + \alpha^2 w(w^Tw)w^T$

$= I + \alpha(\alpha w^Tw - 2)ww^T$ so $\alpha = \frac{2}{w^Tw} = \frac{2}{\|w\|^2}$