Math 5330, Test II (a)

Name Key

Work any 5 problems

1. Given that the QR decomposition of A is

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}, R = \begin{bmatrix} 2 & 5 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$$

use this to find x which minimizes $\|Ax - b\|_2$, where $b = (2, -3, 4)$.

$$Rx \cong Q^T b = \begin{pmatrix} \frac{2}{\sqrt{2}} \\ \frac{3}{\sqrt{3}} \\ \frac{12}{\sqrt{6}} \end{pmatrix} \qquad x = \begin{pmatrix} \frac{5}{2\sqrt{3}} + \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$

2. Find the straight line $f(x) = mx + b$ which most nearly interpolates the points $(0, -1), (2, 2), (3, 3), (5, 4)$ in the least squares sense.

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} \cong \begin{pmatrix} -1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \qquad \begin{pmatrix} 4 & 10 \\ 10 & 38 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 8 \\ 33 \end{pmatrix}$$

$b = -0.5 \quad m = 1$

3. Prove the following:

a. If $A^T Ax = A^T b$, then x minimizes $\|Ax - b\|_2$.

$$\|A(x+e)-b\|^2 = \|Ax-b\|^2 + \|Ae\|^2 + 2(Ae)^T(Ax-b)$$
$$= \|Ax-b\|^2 + \|Ae\|^2 + 2e^T(A^TAx - A^Tb)$$
$$+ 0$$

$$= \|Ax-b\|^2 + \|Ae\|^2 \geq \|Ax-b\|^2$$

b. $I - \frac{2ww^T}{w^Tw}$ is orthogonal, for any vector $w \neq 0$.

$$\left(I - \frac{2ww^T}{w^Tw}\right)^T\left(I - \frac{2ww^T}{w^Tw}\right) = I - \frac{4ww^T}{w^Tw} + \frac{4w(w^Tw)w^T}{(w^Tw)^2} = I$$

4. Find all eigenvalues of the pseudo-triangular matrix

$$\begin{bmatrix} -3 & 7 & 8 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & -4 & 2 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$-3, 8, \frac{3}{2} \pm \frac{\sqrt{47}}{2} i$

5. If the Jacobi iteration $A_{n+1} = Q_n^T A_n Q_n$, where $A_1 = A$, converges to diagonal form in, say, 10 iterations, so that $A_{11} \approx D$, what are the eigenvalues of A, and what are the eigenvectors?

$D \cong A_{11} = Q_{10}^T \cdots Q_1^T A Q_1 \cdots Q_{10}$

$A Q \cong Q D \quad$ where $Q = Q_1 \cdots Q_{10}$

eigenvalues are diagonal elements of D

eigenvectors are columns of Q.

6. a. Find an orthogonal matrix Q such that QAQ^{-1} is upper Hessenberg, if

$$A = \begin{bmatrix} 2 & 4 & -3 \\ 4 & 1 & 7 \\ -3 & 7 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

2

b. Is QAQ^{-1} symmetric (note: you need not actually find A)?

yes

7. a. Do one complete iteration of the LR method, starting with

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -2 & 1 \\ 0 & 4 & -8 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \qquad M_2 M_1 A M_1^{-1} M_2^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 5 & 1 \\ 0 & -48 & -12 \end{bmatrix}$$

b. Is the new matrix still tridiagonal?

yes

c. If you had done a QR iteration instead of LR, would the new matrix still be tridiagonal?

no

Math 5330, Test II 16

Name Key

1. a. Find a QR decomposition of

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 5 & 4.8 \\ 0 & 1.4 \end{bmatrix}$$

2

$$\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4.8 \\ 0 & 1.4 \end{bmatrix}$$

b. Do one complete iteration of the QR method to the matrix A. Is the new matrix more nearly diagonal, in the sense that the sum of squares of the off-diagonal elements is smaller (note: sum of squares of entire matrix will be the same)?

2

$$\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} = \begin{bmatrix} 6.88 & 0.84 \\ 0.84 & 1.12 \end{bmatrix}$$

$(0.84)^2 < (3)^2$ so yes

c. Use the Jacobi method to find all eigenvalue and eigenvectors of A. (Note: only one iteration is necessary!)

3

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda_1 = 7 \quad z_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$

$\lambda_2 = 1 \quad z_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$

2. Prove that if z is a solution to $AA^Tz = b$, then $x \equiv A^Tz$ is a solution of $Ax = b$ of minimum norm.

$Ax = AA^Tz = b$ so x is a solution. Let $Ay = b$ and $e \equiv y - x$, $Ae = Ay - Ax = 0$.

3

$$\|y\|^2 = y^Ty = (x+e)^T(x+e) = \|x\|^2 + \|e\|^2 + 2x^Te$$

but $x^Te = (A^Tz)^Te = z^TAe = z^T0 = 0$ so

$$\|y\|^2 = \|x\|^2 + \|e\|^2 \geq \|x\|^2$$

3. Under certain conditions, the QR iteration produces a quasitriangular matrix in the limit.

   a. Define a quasitriangular matrix. upper Hessenberg, with no two consecutive nonzeros along first subdiagonal

1

   b. In general terms, how do you find the eigenvalues of a quasitriangular matrix?

1 find eigenvalues of 1 by 1 and 2 by 2 diagonal blocks

   c. Is it necessary for convergence, to start the QR iteration from Hessenberg form? What is the advantage of starting from Hessenberg form?

1 not necessary for convergence but decreases work per iteration from $O(N^3)$ to $O(N^2)$

d. If A is symmetric, show that $B = Q^T AQ$ is still symmetric, if Q is an orthogonal matrix. This means that if the original matrix is symmetric, and orthogonal transformations are used to reduce it to upper Hessenberg form, the resulting matrix has what (non-zero) structure? Is $B = M^{-1}AM$ still symmetric, if M is not orthogonal?

1

$$B^T = (Q^TAQ)^T = Q^TA^TQ^{TT} = Q^TAQ = B$$

tridiagonal

$B = M^{-1}AM$ not symmetric

c. If A is upper Hessenberg, the work to do one QR iteration is proportional to what power of N (size of matrix)? What if A is tridiagonal and symmetric? What will happen if the QR iteration is applied to a matrix that is tridiagonal and **not** symmetric?

2

$O(N^2)$ for Hessenberg

$O(N)$ for tridiagonal, symmetric

If tridiagonal and nonsymmetric $A \to$ full upper Hessenberg

3. Consider the iteration $A_{n+1} = AA_n$, where $A_0 = A$, and assume A is diagonalizable ($A = P^{-1}DP$).

a. Show that in the limit as $n \to \infty$, $A_{n+1} = \lambda_1 A_n$, where λ_1 is the largest eigenvalue of A in absolute value (assume there is a largest eigenvalue).

$$A_n = A^n = (P^{-1}DP)^n = P^{-1}D^nP$$

2

$$\frac{1}{\lambda_1^n} A_n = P^{-1}\begin{bmatrix} 1 & & \\ & \left(\frac{\lambda_2}{\lambda_1}\right)^n & \\ & & \ddots \end{bmatrix} P \to P^{-1}EP = F \quad (F \neq 0)$$

so $\dfrac{A_{n+1}}{\lambda_1^{n+1}} \cong \dfrac{A_n}{\lambda_1^n}$ or $A_{n+1} \cong \lambda_1 A_n$

b. The normal power iteration is $v_{n+1} = Av_n$, that is, we normally start with a random vector v_0 and multiply it repeatedly by A, rather than start with the matrix A and multiply it repeatedly by A. What is the advantage of the normal power iteration compared to the alternative approach defined above? Is there any potential disadvantage?

advantage is $O(N^2)$ work per iteration vs $O(N^3)$

may not converge, if $\vec{v}_0$ unlucky choice

$\frac{A^n}{\lambda^n} \not\to 0$ but $\frac{A^n v_0}{\lambda^n}$ may $\to 0$

c. Can you think of a way in which the iteration $A_{n+1} = AA_n$ could be made more efficient ~~than the normal power method~~? (Hint: suppose n is a power of 2)

$$A_0 = A$$
$$A_1 = A^2$$
$$A_2 = (A^2)^2 = A^4$$
$$A_3 = (A^4)^2 = A^8$$
$$\vdots$$
$$A_M = (A_{M-1})^2$$

Math 5330, Test II (c)

Name Key

1. a. Find a QR decomposition of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 12 \\ 0 & -5 \end{bmatrix}$$

3

$$A = \begin{bmatrix} 1 & & \\ & \frac{12}{13} & \frac{5}{13} \\ & -\frac{5}{13} & \frac{12}{13} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 13 \\ 0 & 0 \end{bmatrix}$$

b. Use this QR decomposition to find $min\|Ax - b\|_2$, where $b = (1, 2, -1)$.

2

$$QRx = b$$
$$Rx = Q^T b = \begin{pmatrix} 1 \\ \frac{29}{13} \\ -\frac{2}{13} \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ \frac{29}{169} \end{pmatrix}$$

c. Find $min\|Ax - b\|_2$ using the normal equations method.

3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 12 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 12 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 12 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 169 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix}$$

$$x = \begin{pmatrix} 1 \\ \frac{29}{169} \end{pmatrix}$$

2. (Answer either (a) OR (b), and (c) and (d)). If

$$A = \begin{bmatrix} 12 & 5 \\ 5 & 12 \end{bmatrix}$$

3

a. Do one QR iteration on A.

$$\begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{bmatrix} \begin{bmatrix} 12 & 5 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} \frac{12}{13} & -\frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix} = \frac{1}{169} \begin{bmatrix} 2628 & 595 \\ 595 & 1428 \end{bmatrix}$$

b. Do one LR iteration on A.

$$\begin{bmatrix} 1 & 0 \\ -\frac{5}{12} & 1 \end{bmatrix} \begin{bmatrix} 12 & 5 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{5}{12} & 1 \end{bmatrix} = \begin{bmatrix} \frac{169}{12} & 5 \\ \frac{595}{144} & \frac{119}{12} \end{bmatrix}$$

c. Use the power method to find the largest eigenvalue (in absolute value) of A, starting with $x_0 = (2, 1)$. (Hint: the eigenvalues are integers.)

3

$x_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad x_2 = \begin{pmatrix} 458 \\ 409 \end{pmatrix}$ $\quad x_4 = \begin{pmatrix} 126482 \\ 124081 \end{pmatrix}$

$x_1 = \begin{pmatrix} 29 \\ 22 \end{pmatrix}$ $\quad x_3 = \begin{pmatrix} 7541 \\ 7198 \end{pmatrix}$

16.8

17.2

$\lambda = 17$

d. Repeat (c) but start with $x_0 = (1, -1)$. Explain why the answer is not the same as in (c). What would you expect to happen if you started with $x_0 = (1, -1.001)$?

2

$x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \lambda = 7$

$x_1 = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$

starting w/ eigenvector for wrong eigenvalue

start w. $(1, -1.001)$ will eventually converge to 17.

3. (Answer either (a) or (b)). If

$$A = \begin{bmatrix} 1 & -12 & 5 \\ -12 & 2 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

3

a. Find an orthogonal matrix Q such that QAQ^{-1} is upper Hessenberg.

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{12}{13} & -\frac{5}{13} \\ 0 & \frac{5}{13} & \frac{12}{13} \end{bmatrix}$$

b. Find an elementry matrix M such that MAM^{-1} is upper Hessenberg.

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{12} & 1 \end{bmatrix}$$

4. Find the eigenvalues of the quasitriangular matrix:

$$A = \begin{bmatrix} 2 & -12 & 5 & -7 \\ 0 & 12 & 5 & 3 \\ 0 & 5 & 12 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

3

$\lambda = 2, -4, 7, 17$

Math 5330, Test II (B)

Name Key

1. a. Find $min\|Ax-b\|_2$, using orthogonal reduction, where $b = (1, 1, -1)$ and:

$$A = \begin{bmatrix} 3 & 4 \\ -4 & 3 \\ 0 & 12 \end{bmatrix}$$

4

$$Q_1 = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{13} & \frac{12}{13} \\ 0 & \frac{-12}{13} & \frac{5}{13} \end{bmatrix}$$

$$Q_2 Q_1 A x = Q_2 Q_1 b$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 13 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -\frac{1}{5} \\ -\frac{5}{13} \\ -\frac{109}{65} \end{bmatrix}$$

$$x = -\frac{1}{25}$$
$$y = \frac{-5}{169}$$

b. Find $min\|Ax - b\|_2$ using the normal equations method.

3

$$A^T A x = A^T b$$

$$\begin{pmatrix} 25 & 0 \\ 0 & 169 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

$$x = \frac{-1}{25}$$
$$y = \frac{-5}{169}$$

2. Find the straight line $f(x) = mx + b$ which most nearly interpolates the points $(0, -2), (2, 1), (3, 2), (5, 7)$ in the least squares sense.

4

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} \cong \begin{bmatrix} -2 \\ 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 10 \\ 10 & 38 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 8 \\ 43 \end{bmatrix}$$

$$b = -\frac{63}{26}$$
$$m = \frac{46}{26}$$

3. If

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

a. Do one QR iteration on A.

$$Q = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \qquad QA = \begin{bmatrix} 5 & 4.8 \\ 0 & 1.4 \end{bmatrix}$$

4

$$QAQ^T = \begin{bmatrix} 6.88 & 0.84 \\ 0.84 & 1.12 \end{bmatrix}$$

b. Use the Jacobi method to find all eigenvalues and eigenvectors of A. (Note: only one iteration is necessary!)

$$Q^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad Q^T A Q = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

4

$$\lambda_1 = 7 \quad z_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = 1 \quad z_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

c. Use the power method to find the largest eigenvalue (in absolute value) of A, starting with $x_0 = (2, 1)$.

$$x_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

3

$$x_1 = \begin{pmatrix} 11 \\ 10 \end{pmatrix} \quad \lambda \cong 7$$

$$x_2 = \begin{pmatrix} 74 \\ 73 \end{pmatrix}$$

2

4. a. Show that if A is symmetric, and Q is orthogonal, QAQ^{-1} is still symmetric.

$$B = QAQ^{-1} = QAQ^{T}$$

$$B^{T} = (QAQ^{T})^{T} = QA^{T}Q^{T} = QAQ^{T} = B$$

3

b. Find an orthogonal matrix Q such that QAQ^{-1} is upper Hessenberg (and therefore tridiagonal, since A is symmetric).

$$A = \begin{bmatrix} 1 & -12 & 5 \\ -12 & 2 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{12}{13} & \frac{-5}{13} \\ 0 & \frac{5}{13} & \frac{12}{13} \end{bmatrix}$$

3

5. Find the eigenvalues of the quasitriangular matrix:

$$A = \begin{bmatrix} 2 & -12 & 5 & -7 \\ 0 & 2 & -6 & 3 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$(2-\lambda)(1-\lambda) + 12 = 0$$

$$\lambda^2 - 3\lambda + 14 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9-56}}{2}$$

$$\lambda = \frac{3}{2} \pm \frac{\sqrt{47}}{2} i$$

$$\lambda = 2, -4, \frac{3}{2} \pm \frac{\sqrt{47}}{2} i$$

Math 5330, Test II (e)

Name Key

1. Given that the QR decomposition of A is

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}, R = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

use this to find x which minimizes $\|Ax - b\|_2$, where $b = (2, 3, 1)$.

3

$$Q^T b = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{6}{\sqrt{3}} \\ \frac{-3}{\sqrt{6}} \end{pmatrix} \qquad \left[\begin{array}{cc|c} 1 & -2 & -1/\sqrt{2} \\ 0 & 1 & 6/\sqrt{3} \\ 0 & 0 & -3/\sqrt{6} \end{array}\right]$$

$x_2 = \frac{6}{\sqrt{3}} = 3.46$

$x_1 = \frac{-1}{\sqrt{2}} + 2x_2$

$= \frac{-1}{\sqrt{2}} + \frac{12}{\sqrt{3}}$

$= 6.22$

2. a. Prove that if z is a solution to $AA^T z = b$, then $x \equiv A^T z$ is a solution of $Ax = b$ of minimum norm.

3

$Ax = A(A^T z) = b$ so x is solution

Let $Ay = b$ also, $e \equiv y - x$

$\|y\|^2 = \|x+e\|^2 = (x+e)^T(x+e) = x^T x + e^T e + 2x^T e$

$= \|x\|^2 + \|e\|^2 \geq \|x\|^2$ since

$2x^T e = 2(A^T z)^T e$

$= 2z^T A e = 2z^T(b-b) = 0$

b. Use (a) to find the quadratic polynomial $a + bx + cx^2$, with minimum value of $a^2 + b^2 + c^2$, which passes through the two points $(0, 1), (1, 3)$.

3

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$AA^T z = b$

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x = A^T z = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$1 + x + x^2$

3. If

$$A = \begin{bmatrix} 12 & 5 \\ 5 & 12 \end{bmatrix}$$

a. Do one QR iteration on A.

2

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} 12 & 5 \\ 5 & 12 \end{bmatrix}\begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{pmatrix} 15.55 & 3.52 \\ 3.52 & 8.45 \end{pmatrix}$$

$-12s + 5c = 0$

$s = \frac{5}{13}$

$c = \frac{12}{13}$

$$= \frac{1}{13}\begin{bmatrix} 169 & 120 \\ 0 & 119 \end{bmatrix}\begin{bmatrix} \frac{12}{13} & -\frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}$$

b. Do one LR iteration on A.

2

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}\begin{bmatrix} 12 & 5 \\ 5 & 12 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -r & 1 \end{bmatrix} = \begin{pmatrix} 14.08 & 5 \\ 4.13 & 9.92 \end{pmatrix}$$

$12r + 5 = 0$

$r = -\frac{5}{12}$

$$= \begin{pmatrix} 12 & 5 \\ 0 & \frac{119}{12} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 1 \end{pmatrix}$$

c. Use the Jacobi method to find all eigenvalues and eigenvectors of A. (Note: only one iteration is necessary!)

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{pmatrix} 12 & 5 \\ 5 & 12 \end{pmatrix}\begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{pmatrix} 17 & 0 \\ 0 & 7 \end{pmatrix}$$

2

$c(-12s+5c)+s(-5s+12c)=0$

$5c^2-5s^2=0$

$c=s=\frac{1}{\sqrt{2}}$

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

d. Use the power method to find the largest eigenvalue (in absolute value) of A, starting with $x_0 = (1, 2)$.

$x_1 = Ax_0 = \begin{pmatrix} 22 \\ 29 \end{pmatrix}$ $x_2 = \begin{pmatrix} 409 \\ 458 \end{pmatrix}$ $x_3 = \begin{pmatrix} 7198 \\ 7541 \end{pmatrix}$ $x_4 = \begin{pmatrix} 124081 \\ 126482 \end{pmatrix}$

2

$\frac{124081}{7198} = 17.2$ $\frac{126482}{7541} = 16.8$

$\lambda_1 \simeq 17$

$z_1 \simeq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

4. Find an orthogonal matrix Q such that QAQ^{-1} is tridiagonal, if

$$A = \begin{bmatrix} 2 & 5 & -12 \\ 5 & 1 & 7 \\ -12 & 7 & 1 \end{bmatrix}$$

3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}\begin{bmatrix} 2 & 5 & -12 \\ 5 & 1 & 7 \\ -12 & 7 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

$-5s-12c=0$

$s=\frac{-12}{13}$

$c=\frac{5}{13}$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{13} & \frac{-12}{13} \\ 0 & \frac{12}{13} & \frac{5}{13} \end{bmatrix}$$

Math 5330, Test II (A)

Name Key

1. Given that the QR decomposition of A is

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}, R = \begin{bmatrix} 2 & 5 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$$

use this to find x which minimizes $\|Ax-b\|_2$, where $b = (\sqrt{3}, -\sqrt{3}, \sqrt{3})$.

4

$Rx = Q^T b = \begin{pmatrix} 0 \\ 1 \\ 2\sqrt{2} \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 5 \\ 0 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \cong \begin{pmatrix} 0 \\ 1 \\ 2\sqrt{2} \end{pmatrix}$

$y = -\frac{1}{3}$

$x = \frac{5}{6}$ $\quad \left(\frac{5}{6}, -\frac{1}{3}\right)$

2. Prove that if z is a solution to $AA^Tz = b$, then $x \equiv A^Tz$ is a solution of $Ax = b$ of minimum norm.

$Ax = b$ obviously. If $Ay = b$ also, let $e \equiv y - x$

4

$\|y\|^2 = \|x+e\|^2 = x^Tx + 2x^Te + e^Te = \|x\|^2 + \|e\|^2 + 2x^Te$

$= \|x\|^2 + \|e\|^2 + 2(A^Tz)^Te = \|x\|^2 + \|e\|^2 + 2z^T\overset{0}{Ae}$

$= \|x\|^2 + \|e\|^2$

$\geq \|x\|^2$

3. Find all eigenvalues of the pseudo-triangular matrix

$$\begin{bmatrix} 1 & 3 & 8 & -1 \\ -4 & 2 & 3 & 4 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

4

$(1-\lambda)(2-\lambda) + 12 = 0$ $\quad \lambda = \frac{3 \pm \sqrt{9-56}}{2} = \frac{3 \pm \sqrt{47}\,i}{2}$

$\lambda^2 - 3\lambda + 14 = 0$

$\left\{\frac{3 \pm \sqrt{47}\,i}{2}, 2, 11\right\}$

4. If the Jacobi iteration $A_{n+1} = Q_n^T A_n Q_n$, where $A_1 = A$ converges to diagonal form in, say, 5 iterations, so that $A_6 \approx D$, what are the eigenvalues of A, and what are the eigenvectors?

4 D contains eigenvalues $Q_1 Q_2 Q_3 Q_4 Q_5$ contain eigenvectors in columns

5. If

$$A = \begin{bmatrix} 2 & 12 & -5 \\ 12 & 1 & 7 \\ -5 & 7 & 1 \end{bmatrix}$$

a. Find an orthogonal matrix Q such that QAQ^{-1} is upper Hessenberg.

3

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{12}{13} & \frac{-5}{13} \\ 0 & \frac{5}{13} & \frac{12}{13} \end{bmatrix}$$

b. Find an elementary matrix M such that MAM^{-1} is upper Hessenberg.

3

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{12} & 1 \end{bmatrix}$$

6. Do one complete iteration of the LR method, starting with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -2 & 1 \\ 0 & 4 & -8 \end{bmatrix}$$

$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$

4

$M_2 M_1 A M_1^{-1} M_2^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 5 & 1 \\ 0 & -48 & -12 \end{bmatrix}$

7. Use the power method to find the largest (in absolute value) eigenvalue of

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 10 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Start with $(1, 5, 1)$ and do 3 iterations. What is the corresponding eigenvector?

4

$x_0 = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$ $x_1 = \begin{pmatrix} 6 \\ 52 \\ 6 \end{pmatrix}$ $x_2 = \begin{pmatrix} 58 \\ 512 \\ 58 \end{pmatrix}$ $x_3 = \begin{pmatrix} 590 \\ 5436 \\ 590 \end{pmatrix}$

10,2

10.6

$\lambda \simeq 10.2$ $z = \begin{pmatrix} 0,11 \\ 1 \\ 0,11 \end{pmatrix}$

Math 5330, Test II (g)

Name Key

1. a. In problem 2.9, you proved that if z is a solution to $AA^Tz = b$, then $x \equiv A^Tz$ is a solution of $Ax = b$ of minimum norm. Use this to find the quadratic polynomial $a + bx + cx^2$, with minimum $a^2 + b^2 + c^2$, which passes through the two points $(0, 2), (1, 3)$.

3

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad AA^Tz = b \qquad \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$z_1 = 1.5 \quad z_2 = 0.5$

$$x = A^Tz = \begin{pmatrix} 2 \\ 0.5 \\ 0.5 \end{pmatrix} \qquad 2 + \tfrac{1}{2}x + \tfrac{1}{2}x^2$$

b. Find the least squares quadratic polynomial fit, $a + bx + cx^2$, to the points $(0, 2), (1, 3), (2, 2), (3, 2)$. Use the normal equations, but you do not need to solve the final linear system.

2

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad A^TAx = A^Tb$$

$$\begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 29 \end{pmatrix}$$

$$x = \begin{pmatrix} 2.15 \\ 0.65 \\ -0.25 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \end{pmatrix}$$

2. What is the order of work (power of N) for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

4

a. The Jacobi method to find the eigenvalues of a symmetric matrix A.

$O(N^3)$

b. One LR iteration, if A is upper Hessenberg (assume no pivoting). $O(N^2)$

c. One QR iteration, if A is symmetric and tridiagonal. $O(N)$

d. One power method iteration. $O(N^2)$

e. The first inverse power method iteration. $O(N^3)$

f. The second inverse power method iteration, assuming the LU decomposition from the first iteration is saved. $O(N^2)$

g. The orthogonal transformation of a full matrix to a similar upper Hessenberg matrix. $O(N^3)$

3. Explain how you would find the vector x which minimizes $\|Ax - b\|_2$, if you already have the QR decomposition of the M by N matrix A. The operation count would be $O(N^\alpha)$ for what α, if we assume $M \approx 2N$? What would the operation count be if you don't have a QR decomposition?

$QRx \approx b \rightarrow Rx \approx Q^T b$

2

multiply b by Q^T, solve upper-triangular system $Rx = Q^T b$ ignoring zero rows. $O(M^2) + O(N^2) = O(N^2)$

IF no QR decomposition $O(MN^2) = O(N^3)$

4. If

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

a. Use the Jacobi method to find all eigenvalues of the matrix. Only one iteration is necessary!

3

$$\begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix} \begin{pmatrix} 5 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 5 \end{pmatrix} \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} 5+6cs & 0 & 3c^2-3s^2 \\ 0 & -1 & 0 \\ 3c^2-3s^2 & 0 & 5-6cs \end{bmatrix}$$

$3c^2 - 3s^2 = 0$
$c^2 + s^2 = 1$ $\Rightarrow c = s = \frac{1}{\sqrt{2}}$

$$\Rightarrow QAQ^T = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda = 8, -1, 2$

b. Find an orthogonal matrix Q such that QAQ^T is upper Hessenberg (and quasi-triangular).

2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

c. Now find the eigenvalues of the quasi-triangular matrix resulting from part (b).

2

$$\det \begin{pmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix} = -(\lambda+1)(\lambda-8)(\lambda-2) = 0$$

$$\lambda = -1, 2, 8$$

d. Do 2 or 3 iterations of the power method to find the largest eigenvalue (in absolute value) of A, starting with $x_0 = (2, 0, 3)$.

2

$$x_0 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \quad x_1 = \begin{pmatrix} 19 \\ 0 \\ 21 \end{pmatrix} \quad x_2 = \begin{pmatrix} 158 \\ 0 \\ 162 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1276 \\ 0 \\ 1284 \end{pmatrix}$$

$$\lambda \cong \frac{1276}{158} = 8.1$$

$$\cong \frac{1284}{162} = 7.9$$

$$\lambda \cong 8$$